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PLANE GLOBE PROJECTION -

A LINNEAN SYSTEM OF MAP PROJECTION

[Translated from Min.-Rat. Prof. Dr. HANS MAURER'S "EBENE KUGELBILDER," (Ergängungsheft Nr. 221 zu "Petermanns Mitteilungen") Gotha, 1935.]

Foreword and Editing
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FOREWARD

The work at hand concerns classification of geographical map projections and is an edited though still quite literal translation of Min.-Rat. Prof. Dr. Hans Maurer's "Ebene Kugelbilder -- Ein Linnesches System der Kartenentwurfe," Petermanns Mitteilungen, Ergangungsheft Nr. 221, Goth, 1935. The editing has consisted principally of putting text, maps, tables and charts into a format that could be accomodated in a vehicle of publications to which such advantages as multi-color representation and fold-outs are denied. The literalness of the translation will be apparent immediately, but hopefully not painfully, to all but those totally untutored in matters of cartography. Instead of finding such familiar terms as "conformality" and "equivalence", one will see instead "shape-true" and "area-true", respectively. Other literalisms also have been permitted to exist.

Maurer's paper was written during the 1930's, a period when Germans were attempting to "purify" and "germanize" -- among other things -- their language. Our translator, Peter Ludwig, has attempted, as noted above, to render nothing more than a "direct", rather literal translation. Thus, the Latin and Greek derivative specialized terms usually used are often avoided. Cartographers will notice this as soon as they see the names for the classification categories. But Herr Maurer laid great store by his germanizations and made fine distinctions among various properties with them. See

his remarks about this matter on page 13. Herr Maurer's list of "germanizations" has been rearranged so as to aid the reader of this translation. (See p. 188)

Mathematical expressions that have been "lifted" from the text have no change except for the German decimal-comma, which has been changed to decimal-point. In tables that have been reproduced directly, however, the decimal-comma stands. These and other differences in symbols and punctuation should cause little difficulty, however.

Classification in a discipline is, of course, not an end, but rather the means to an end. Success, however, can be denied unmitigatedly, to those who labor with inadequate or confused classifications. To geographers who would study the reasons for various spatial patterns of phenomena on the earth's surface the map is at once a means of storing information, experimenting with it, solving various spatial-type problems, and presenting certain results of analysis and synthesis. The need, therefore, to delineate and group map projections for various purposes according to various characteristics seems self-evident.

Perhaps it will not be overly repetitious to note yet another time that the earth's surface when regarded as the surface of a sphere is not developable to the plane. It cannot be spread out in a plane without stretching, overlapping or tearing. Even for an infinitesimal element of area on the earth's surface, the plane

map must, in general be distorted either in shape or in area (even apart from the scale factor). It may, of course, be "distorted" in both area and shape. If, however, within each infinitesimal element of area, there is no "distortion" of shape, the map is said to be conformal. If there is no distortion of area, the map is said to be equi-areal or equivalent. This is achievable because linear scales along orthogonal lines may be compensated as required.

Equivalence is an overall property whereas conformality can not be because isometric mapping (i.e., everywhere constant linear scale) is not achievable. Whereas every local angle on the Greenland coastline can be preserved, the overall shape can not be. Greenland just does not possess a plane shape.

Geodesic preservation can be achieved, but only at the expense of both equivalence and conformality. It is possible to preserve conformality and obtain other additional features, however. For example, a rhumb line which is a constant bearing line on the earth's surface (not generally a great circle) may be shown as a straight line and with conformality preserved.

The term "distortion" is frequently used in descriptions of properties of map projections. If the projection is truly a systematic, mathematically defined one, the term "transformation" or some such less pejorative notion is preferable. One should not say, for example, that angles are distorted on an equivalent

map. Angles are recoverable if the function is known, though they may not be successfully measured directly on the map in a manner analogous to that on the earth. The key to the matter seems to be that we regard as "preserved" those properties for which the same methods of measurement may be employed on both the map and the earth with identical results and as "distorted" those for which identical methods produce different results, requiring thus, different methods to achieve comparable values.

Other matters pertinent to projection such as the nature and location of necessary interruptions (whether of point, line, or area, or in combination) in mapping from the closed surface of a sphere to the open surface of a plane, singularities, and many other topics also map be considered as varying aspects of projections.

It can be seen that families of projections may be regarded as existing depending upon which property or groups of properties the classifier wishes to consider. The infinite variety of map projections possible and the general lack of mutual exclusiveness of properties (a notable exception being, as noted, conformality versus equivalence) renders the problem of classification difficult. In fact, the problem of classification of map projections can stand as a classic example of the "taxonomic" difficulties encountered in constructing meaningful classes in general. The value of any classification can be judged only against the use to which it

is to be put. But, since projections are defined by mathematical functions, system can be introduced into the classification procedure.

In "A Classification of Map Projections," Annals of the Association of American Geographers," Vol. 52, 1962, pp. 167-175, Professor Waldo Tobler (Department of Geography, University of Michigan) notes that the undoubtedly increased use of electronic computers and quantitative methods on the part of geographers in the future will require a more detailed understanding of map projections than has been the case before this. Anticipating this need, Tobler, himself, has created a "parametric" classification that provides a simple direct approach to the problem. It is intermediate between the verbal-descriptive categorizations ordinarily found in geography text books and a truly general classification based, for example, on complex variables or vectors, as Tobler notes. The Tobler classification does include the normal cases of all possible map projections, however. In essence Tobler's classification explores the possibilities for mapping spherical coordinate systems to plane coordinates when either or both plane coordinates depend upon either or both spherical coordinates according to various functions.

The figure, from the Tobler article, reproduced below neatly summarizes his system.

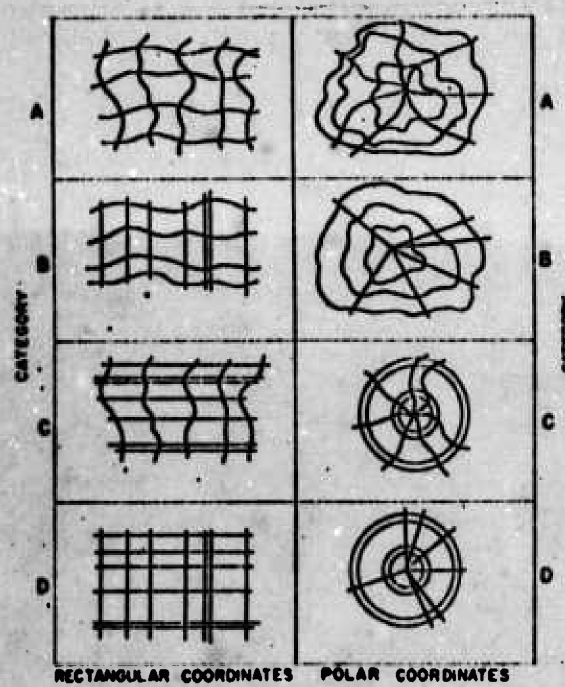

FIG. 1. Schematic illustration of map projections in the classificatory system. The diagrams do not represent any specific projections but are indications of the possible form of the lines of latitude and longitude on the map. In category A both families of parametric lines may be variably spaced curves. In category B the meridians may be variably spaced straight lines (parallel or radiating, depending on the coordinates employed). The parallels may be variably spaced curves of highly variable curvature. Category C is similar to category B except that the role of the meridians and parallels are interchanged; the parallels in polar coordinates are circles of course. The forms of category D are familiar; the spacing illustrated is more variable than that usually encountered, however. Interruption, similar to that on conic projections, and truncation have not been illustrated but also are possible. The diagrams all refer to the normal cases of the projections.

Of considerable importance is the fact that the system includes various new "relative" mappings so much appreciated by theoretical geographers wherin "distance" is measured in time, cost, effort, or in other pertinent terms.

As we continue our studies of the role of distance as a dimension of society, we are constantly reminded that its importance is to be judged not in physical units of length alone, but rather in terms of "cost distances", "time distances" and the like. We are in-

formed of the concept of the elastic mile. The straight line distance over the earth's surface is not, we are told, to be regarded necessarily, as the effective distance separating places. Rather, circuitous land routings between places, for example, may, within limits, prove more economical if intervening difficult terrain and the high cost of traversing it can thus be avoided. Then, too, the favorable ratio of water transport rates to land transport rates frequently occasion a willingness to add substantial amounts to the geographical distance involved, the result being significant economic gain.

To man, operational space on the earth's surface is not at any one moment to be conceived as representing a single geometric unity. For example, many conceptual surfaces may be regarded as overlying the physical surface of the earth, each such conceptual surface being defined operationally in its appropriate terms. Such surfaces co-exist but do not coincide. Their reconciliations present problems of major significance to planners and modern scholars of society. One man's geodesics are another man's crooked lines, just as one man's data are another man's trivia.

The Tobler system is general enough to include certain "empirical" projections as well. In his paper, however, Tobler mentions specifically, as examples, only a few "named" projections, his emphasis being, and rightly so, on the development of the system and presenting it before any exhaustive application to all

of past cartographic history was made. Others who recently have concentrated as well on the development of classification systems without extensive applications include L.P. Lee in "The Nomenclature and Classification of Map Projections," Empire Survey Review, Vol. 7, 1944, pp. 190-200 and B. Goussinky in "On the Classification of Map Projections," in Vol. 11, 1951 of the same journal, pp. 75-79.

Indeed, in the English Language, no single work exists that can be regarded as at once a detailed catalogue of named projections and an incorporation of aspects of cartographic history according to systematic principles. The British National Committee for Geography of The Royal Society have issued a "Glossary of Technical Terms in Cartography," London, 1966 containing in Appendix I an alphabetized listing and very brief verbal descriptions of about 170 named projections. They have indicated the number assigned to a projection in the Maurer catalogue where that is pertinent. The British listing is not as inclusive as the Maurer catalogue for entries until 1935, but does include more recent ones and is valuable in that respect.

In order to help fill this serious gap in the literature for English speaking geographic and cartographic students the following rendition of Maurer's work is offered. It is hoped that this entry in the Harvard Papers in Theoretical Geography will provide not only a source of valuable information to established scholars

but especially that it will also provide an introduction to the nature of map projections, the history of cartography, and the essence and problems of classification of projections for beginning graduate students

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SECTION I: INTRODUCTION

Prefatory Remarks

The purpose of this paper is to bring better order and clarity into the diversity of the various systems of map projection. The projections will be ordered according to stems, classes, families, etc., in which various common attributes are to be seen. The first classical example of such an attempt at classification was in another area of very intricate diversity, the system of classification in the world of plants and animals, as drawn up by Linne. Then came other systems of classification, according to other criteria, such as the classical system of climate grouping drawn up by Koppen.

Now, of course, map projections are not an assortment of phenomena offered by nature, but rather they have been thought up by human beings and can be multiplied at a quite arbitrary rate. Nevertheless, even for these one can find principle criteria, which can bring too often missed order and clarity to the field of cartography. It is most necessary that the characteristics, which are supposed to be peculiar to the various groups and which belong to the terminology of cartography, be uniformly, clearly and consistently established. Until now, this has been missing to an alarming extent in the development of this science. There have even been well-known cartographers who have taken the position that truly sharp, consistent and concise definition of the criteria for map projection is not necessary, since any conflicts because of inexact definition would be made hardly recognizable in the types of projection found in the geographical atlases. On the other hand, this paper will defend the position which I have made known in the past:

"The above mentioned view, which is the source of the chaos in the system of designation in cartography, is in my opinion completely uncalled for. Theoretical cartography, the theory in respect to illustration of the earth's surface on plane surfaces, is fundamentally a mathematical science and cannot do without clear basic terminology. One does not need to become entangled in mathematical sophistries, but inexactitude does not need to have such free reign that even in the best texts one cannot tell what is meant by the various illustrations." On the other side, many people have regretted the present chaos; as evidence of this I shall confine myself to repeating the statement of Eckert[1] in his great work "The Science of Cartography" ("Die Kartenwissenschaft"): "In the nomenclature of projections there reigns a muddledness which is not to be found in any other branch of geography." Giving names obviously has the purpose - or, at least, should have - of making clear the particular characteristics of known things. Therefore I have attempted, not only to classify projections in one large system, but for the projections themselves, to suggest and to use appropriate German terms for their particular characteristics and for any other designations which arise in the field of cartography.

I made known the principal features of the system many years ago in a lecture to cartographers of Berlin, but it is only in retirement that I have found time to work it out quite thoroughly. The careful reader will discover at appropriate places in this paper, now and then with surprise, that this systematically comparing research of the types of projection will lead to the tracing down and the correction of a great number of blunders and errors in the text books of our field. The reader will find in my system many types of projection which I have added, aside from those known from our text books and which I name with reference to their sources. I must, in this respect, ward off the likely reproach that I have suggested a huge number of useless projections. The numbers

of the tables of the system are in general not proposals for atlases, but rather they have been set up for the completion, for the elucidation of the relationships between other types of illustration, or for the explanation of the consolidation of some characteristics into one projection. This is true for practically the whole Stem IV of system tables and for the system numbers 30 - 33, 35, 36, 41, 42, 54 - 58, 60 - 62, 74 - 76, 80, 96 - 99, 101 - 106, 157, 162, 164, 165, 168, 195. Another group presents generalizations and different, justified special cases in respect to projections in the text books; this is true of system numbers 5, 17, 19, 20, 43, 86, 110, 113, 115 - 117, 119, 121 - 124, 126, 127, 130, 138 - 140, 142, 144, 145, 147, 148, 150, 151, 171. And only those numbers which I have entered under my own name in the "Index of Types of Projection" ("Verzeichnis von Entwurfsarten") can be considered as proposed projections, which come into question for special designs.

Illustrations from only a small part of the many projections are included in this paper, although, as far as well-known types of projection are concerned, they are referred to in the column labeled literature in the system tables; especially those with the designation F refer to the number of the figure in the "Manual of Map Projection" ("Leitfaden der Kartenentwurfslehre") by Zöppritz - Bludau, Leipzig and Berlin, 1912.

The Map Grids as Characteristic of the Projections

First of all, one must decide just which clearly defineable characteristics of map projections are to form the foundation of their classification. If we understand a map of the world to be a two dimensional picture of the earth's surface, then the map must be of such a likeness, that every single point of the earth's surface be recognizeable. For

this, we need a system of coordinates on the earth itself, according to which we can indicate the original point and a system of coordinates on the map, thus enabling ourselves to locate the illustration in relationship to the illustrated. Thus the science of cartography becomes the science of the relationships between a grid of coordinates on the earth and a corresponding grid of coordinates in the various possible two-dimensional representations. We want to perceive the original, which is to be represented, as always lying on a globe, the radius of which is equal to the unit of length; in other words, we want to consider - or, to be more exact, to assume - that a picture of the earth has first been projected onto the above mentioned uniform globe, and then that the map projection represents this picture in two dimensions. *What we see, then, as characteristic of a map projection, is its grid of coordinates, which is the likeness of a grid of circles on the uniform globe.*

Now, the grid of meridians and parallels of latitude is the usual grid on the uniform globe; its illustration is supposed to be called the *polar grid*. But since all points and all great circles of the globe, in view of their geometric traits, are completely equal, we can instead of the meridians and parallels of latitude - that is to say, the great circles which cut through the poles and the secondary circles which cut through these great circles - choose just as well a similar grid of prime and secondary circles of two other arbitrary antipodal points. We have at our disposal on the globe so infinitely many coordinate grids of the exact same shape. In a given two dimensional representation, however, the likenesses of all these grids are by no means of the exact same configuration and they show definitely dissimilar geometric characteristics. Of course, there are special characteristics, which, independent of the coordinate grid drawn into the map, apply to every point or every piece of area of the map and can serve as distinguishing mark of

the particular type of representation. Such characteristics will be defined by means of differential equations, which are valid for the whole map. Thus it happens in fact for the valuable attributes of *true-shape* (Winkeltreue or conformality, tr.), on the one hand, and *true-area* (Flächentreue or equivalence) on the other. As a matter of fact, the mathematician tends to include all conformal and all true-area projections in one main group. In cartography, however, one has not done this, and no doubt correctly, precisely because the third main group of the neither true-shape nor true-area projections consists of such a great number, which have to be classified according to completely different viewpoints. But it is above all these dissimilar forms of grids, whose particular properties are ascertained by means of differential equations, which are connected with the just mentioned group. For this reason one prefers laws of classification, according to which the similarity among the principle forms of grids, stands out clearly and graphically. Therefore, we also use true-shape and true-area, not as the marks of the principle families of the system, but rather only for the smaller subdivisions. Yet we must then relate in general the characterizing attributes of a projection to a particular grid of coordinates. Thus, *our system of map projections becomes the same as a system of map grids.*

According to the rules, one will have the polar grid considered as the grid of a projection, if another grid of prime- and secondary-circles of the globe does not produce essentially greater geometric simplicity. One must maintain the polar grid as the distinguishing grid, however, particularly in the event that the polar grid is involved in the meaning and use of the map. This is, for example, the case with respect to true-course (Loxodromen) maps, on which the lines of course remain the same, or with respect to equal-azimuth maps, on which the lines are straight lines. For the terms *course* and *azimuth* are connected inseparably with the meridians. Since

our system is to have general application, however, we may not fix the polar grid as the only valid grid for projection, but rather we must define as applicable the circle grid with two definite, although arbitrarily chosen, antipodal points of the globe; for the time being this grid will be considered the *basic grid* of this plan.

The Grid Lines

We shall employ, then, two hosts of circles as coordinate grid of the globe, to be more exact, the *prime circles*, great circle halves, which have two points on a diameter of the globe in common, and the *secondary circles*, full circles which intersect all these principle circles perpendicularly. The two points which are common to all prime circles are called the *prime global points*, and the connecting straight line, the *prime axis*. We consider the degree marking for the prime circles determined in this manner: A distribution of degrees runs in both directions from a *central prime circle* around the prime points as center, and each prime circle is designated by the angle λ (between $-\pi$ and $+\pi$), i.e., the angle formed by each prime circle with the *central prime circle*.

The largest secondary circle is called the *basic circle*. The degree marking of the secondary circles is given by means of a distribution of angles formed with the principle circles, which runs, either in both directions from the basic circle as $\pm\phi$ up to $\pi/2$ to the prime points, or from one prime point, the *basic point* as δ from 0 up to π to the other prime point. Each secondary circle is designated by the value ϕ or δ, which applies for the point of intersection with the prime circles.

The prime and secondary circles are considered as set off by equally spaced values of λ and ϕ (and/or δ), a grid

of these curves thereby resulting. Between each two prime circles of the grid lies a *cell*(Fach) (principle circle - intersection), and between each two neighboring secondary circles lies a *zone* of the grid.

In the representation of this globe grid, of this *map grid*, the representations of prime circles are called *prime lines* (Hauptlinien) (abbreviation: H); the pictures of the secondary circles are called *secondary lines* (Nebenlinien) (abbreviation: N). The terms *cell* and *zone* are also valid in the same sense for the map grid. If the uniform division of a grid line on the globe corresponds with a uniform division of the line as illustrated, then this is considered *equally-divided* (Gleichteilig or uniform).

Absolute dimensions on a map are of consequence for the types of representation and their rules only when the projection is supposed to be true-area or distance-interval-true (Abstandstreue or equidistant). As already mentioned, we assume that the radius of the globe which is to be illustrated represents the unit of length. True-area means, than, that all units of area in the picture correspond with the area in the original picture on the uniform globe. *Distance-interval-true* means for a pair of map points that the shortest distance between them (geradliniger Abstand) is exactly corresponding to the length of that arc of the great circle, as it is between its two original points on the globe. Such accuracy with respect to distance interval cannot possibly be fulfilled for all pairs of points on a map. One must therefore *qualify* explicitly, *when one calls a whole projection true with respect to distance interval*. By such projections we mean those on which the N are either equal-centered (concentric) circles or parallel curves, and the intervals between the circles or the parallel curves correspond to the intervals of the great circle arcs between the original secondary circles on the globe. Such a projection

is called *central-interval-true* (mittabstandtreu), if there is a point on the map from which the intervals between all other map points can be reproduced on an accurate scale; if such a point is missing, then the projection with interval accuracy is *circle-interval-true* (kreisabstandstreu).

The Grid Coordinates

The illustration is mathematically determined, if the two coordinates of each point of the illustration, which serve either as polar coordinates r, α or rectangular coordinates x,y*), are given as functions of the two original illustration coordinates λ and δ (or ϕ). In general, two representation-equations (Abbildungs-Gleichungen) (abbreviation: Gl.) serve here. According to these equations, a point on the surface of the map does not always have to correspond to a point on the globe, and vice versa. It does happen often, however, that one or two particular points of the globe are represented on the two dimensional surface by a line which can be the infinitely distant straight line. Furthermore, it can occur that according to the representation-equation, not only one, but several likenesses of the surface of the globe are produced on the two dimensional representation. This can be related for example with the fact that each global point $[\lambda;\phi]$ also can be considered as $[(\lambda + 2n\pi):(\phi + 2m\pi)]$ or $[(\lambda + (2n + 1)\pi; ((2m + 1)\pi - \phi]$ where m and n are whole numbers and quite different coordinate values and points on the two dimensional representation can correspond to the various global coordinates. Inversely, it is also possible that two or more different global points correspond with one point on the two dimensional representation, so that the

*Note: We chose the x-axis in the direction of the straight line representing central H, the perpendicular to that the y-axis, in accordance with the papers of Tissot-Hammer and Bourgeois-Fürtwängler. Herz and Zöppritz do it the other way around.

same section on the plane surface is overlapped by several sections of the global representation. Naturally, one draws only one of these images on the appropriate section of the two dimensional illustration.

The Inclination of the Projection

For the type of representation as such, it does not matter how the global coordinate grid lies in relationship to the bodies of land and the oceans; but one names the projection *according to its inclination:*

earth-axical (erdachsig), if the prime axis of the global grid coincides with the axis of the earth,

transverse-axical (querachsig), if the prime axis of the global grid is on the plane of the equator, and

oblique-axical (schiefachsig), if the attitude of the prime axis of the global grid corresponds with neither of the two named above.

With a polar inclination the grid of prime and secondary lines coincides with a polar grid, and does not do so with transverse and oblique inclinations. As mentioned above, one may consider every map, which according to its particular type of projection produces a grid not corresponding to the polar grid, as a representation of the polar grid, but just in another manner of projection. As a rule, however, one chooses a map of a particular type of projection, because it surpasses another in regularity and simplicity. It serves as an explanatory example, that with a transverse (orthographic) projection the N become parallel lines in irregular intervals and with irregular distribution, and the H are ellipses. One could consider this map projection a so-called untrue cyclindrical projection (einen säulenkreisigne Entwurf, or conventional cylindrical projection) as does, in fact,

Tissot-Hammer. In such a case, the grid of prime and secondary circles of an equatorial point corresponds to an azimuthal projection of greatest uniformity, with straight true-shape intersecting H and with equally distributed full circles with the same center as N. One would designate such a map, naturally, as an equatorial azimuthal projection of the most simple type, as a perspective representation of the globe, not as a projection on a cylinder according to its polar grid; in the latter case, it would be awkward and unclear to indicate how the representation could be produced on that cylinder, the axis of which is the same as the globe's, and the radius of which could be arbitrary.

The designations *earth-axical* (erdachsig), *transverse-axical* (querachsig) and *oblique-axical* are to be recommended over others used, since they show just what is meant and cannot be misunderstood, as is the case with respect to most other terms used in the place of these. Some of these other terms are:

for *earth-axical*: Perpendicular, pole-oriented, pole-axical, right-axical, polar-projection, equator-projection (normal, polständig, polachsig, rechtachsig, Polar-Projection, Äquator-Projection);

for *transverse-axical*: transversal, equator-oriented, equator-exical, meridian-projection, equator-projection (transversal, äquatorständig, äquatorachsig, Meridian-Projection, Äquatorial-Projektion);

for *oblique-axical*: horizontal, zenithal, inter-oriented, slanting-axical, Meridian-projection, transversal, transverse-axical (horizontal, zenital, zwischenständig, schrägachsig, Meridian-Projektion, transversal, querachsig).

It is obvious, that one will only with difficulty understand the difference between equator- and equatorial-projection.

It is a doubtful matter, when the designations *meridian-projection* and *transversal* apply also for the oblique-axical projections in the pamphlet by Graf-Groll[3] published for a wider circle of readers, and when E. Hammer[4] labels oblique-axical projections also as transverse-axical, just because in both of them a coordinate-axis lies across, i.e., perpendicular to, the straight central-meridian (geradlinigen Mittelmeridian). The word *axis* in the term *transverse-axical* means something completely different from what it means in the word "oblique-axical" as used by Hammer. In view of the unsuitableness of other terms, I call attention to my earlier statements[5] and confine myself here to the following sentences:

If the axis of the projection is perpendicular (*normal*) to the axis of the earth, then the projection is *not* called perpendicular, but transversal, although a transversal really does not need to run perpendicular to that straight line which it is supposed to intersect. If the axis of a projection is not perpendicular to the axis of the earth, but intersects it, then one calls the projection perpendicular. It is called *zenithal* or *horizontal*, if the axis of the projection passes through the zenith of a point of the earth or stands perpendicular on its horizon; but is should not be called horizontal or zenithal, if that point of the earth falls on the equator or on a pole of the earth, as though such points had no zenith and no horizon. A cylindrical projection should be called *equator-oriented*, if the cylinder used does not stand on the equator, and *equator-axical*, if the axis of the projection is not the axis of the equator-circle, i.e. the axis of the earth, but rather any diameter of the equator. A perspective projection should be called a meridian-projection, if the eyelevel meets the level of the equator, but equator-projection when the eyelevel meets the axis of the earth.

All these remarks show how easy it is to misunderstand most terms other than the designations earth-axical, transverse-axical and oblique-axical.

The Grid Lines as Basis for Classification

For classifying, our system of map projections uses mainly the most important properties of the H- and N-lines of the particular projections as characteristics of families, orders, classes, etc. The N and H play a role in our system similar to the stamens and pistils in the classification of flowering plants in Linné's system. The setting up of such a system has not only the advantage that one can get a clear idea about the usefulness of the various projections and about how they are more or less related to each other. Most important, it forces one to sharply define the distinguishing properties and to thereby combat the chaos in the basic terminology of the field of map projection. This extremely deplorable chaos has resulted from the various authors who often use the same designation for competely different concepts or even contradict themselves with their own definitions. My countless attempts to establish consistent and uniform definitions were not mathematical hair splitting with which geography need not meddle, but rather a challenge which every science must meet if it does not want to degenerate.

The system, including the detailed examples of projections, has been carried through so that it can provide for the locating of particular map projections according to their properties, just as a guide to flora facilitates plant classification. It is assumed that the exact description of these particular projections is for the most part known; otherwise, one may use the bibliography given at the end of this paper for investigating. There are also numerous bibliographical references in the systems tables.

The Designations for the Properties of Projections

German terms were used for these designations wherever possible, and germanizations for many of the foreign terms have been suggested. (An alphabetical index of these germanizations is appended on page 188 .) The justification for such germanizing is found not only in the self-evident endeavor to write German wherever possible, but also because occasionally various authors use the same word of foreign origin for different meanings, because for any one concept several designations of foreign origin exist, or finally because it is natural to wish to find a more appropriate term than the completely misleading or meaningless designation of foreign origin. The faultiness of the terms "normal, transversal, horizontal," etc., for the orientation of the projections has already been discussed. The remaining terms and designation will now be more exactly clarified, as they appear on the basis of the system of the great system tables.

The Families (Stamme) of the System)

Both families I and II have the property of being *centrally circular* (Mittelkreisigkeit or axial) in common. A projection is *centrally circular* when all its N are center-equal (concentric) circumferences, whereby circumferences can be not only understood as circle-arcs and full circles, but also as the border cases, point and straight line. I used to use the term "axial" for centrally-circular, but prefer the appropriate German term "centrally circular." The mathematical equation for it is $r = F(\delta)$ independent of λ, when r means the radius of N. If S, the midpoint of the center-equal circumference lies in infinity, then all N are genuine curved circumferences and do not degenerate into straight lines. Such projections make up the Family I denote with the designation "true-circular." If S is infinitely distant then all N become parallel straight lines.

Such projections are called *cylinder-circular* (saulenkreisig or conventional cylinder). For their mathematical reproduction rectangular coordinates are advised, where s = F(ϕ) independent of λ. With almost all cylinder-circular projections, the base-lines $x = \phi = 0$ as well as the mid-H, which is reproduced as a straight line, show up as symmetry-line of the grid (even if this is actually not unavoidable; c.f., no. 197, 198, 207-216, 224 of the system). One can imagine such a projection drawn on a straight circle-cylinder (Kreiszylinder), a *cylinder* (Saule), the axis of which is parallel to the prime axis of the globe and to the above mentioned mid-H, while the straight N intersects cylinder circumferences. The designation "cylinder-circular" is hereby justified. In the literature these cylinder-circular projections are usually called true- and untrue-cylindrical projections (echte and unechte Zylinderprojektionen or conventional cylindrical). While cylinder-circular projection, which do without one or both of the above mentioned symmetries, have only theoretical value but no practical value, the cylinder-circular projections, which have this double-symmetry, play an important role. They represent Family II of our system.

The double-symmetry, which is characteristic of Family II and is missing in Family I, is also possible, when the N are not parallel straight lines, but, rather, curved lines. All projections of this nature are included in Family III. Families I and II, then, are both *centrally-circular* which Family III is not; and Family II and Family III are both *double-symmetrical*, which Family I is not. Our three most important families, then, are:

Family I: True-circular. All N are center-equal circumferences with midpoints in infinity. The projections are not double-symmetrical, because they are not symmetrical to the base-line. If all the N are full circles, the projection is

called radial-circular (radkreisig or azimuthal); if all the N are not full circles, the projection is called "conical-circular" (kegelkreisig or conventional conical).

Family II: Straight-symmetrical. All N are parallel straight lines, and the grid symmetrical to the base-line $\phi = 0$, as well as to the mid-H $\lambda = 0$.

Family III: Curved-symmetrical (Krummspiegelig). The grid is symmetrical to the base-line $\phi = 0$ as well as to the mid-H $\lambda = 0$. The N are, however, not parallel straight lines.

Family IV: consists of the *less regular* projections, which are neither centrally-circular (mittkreisig or achsial) nor double-symmetrical. The uniformity of these grids is mostly limited to symmetry to one straight-line mid-H. They can even be conventional-cylinder, but then without an H as symmetry-line.

Finally, we need yet a *Family V* which constitutes those map projections which are not constructed according to a uniform grid for the whole map but for which different laws of representation or equations are applied to the grid lines on various parts of the world map. This is the *Family of compound projections*.

It should be carefully noted that our property of classification *"double-symmetrical"* does not mean that the grid projection must be symmetrical to any two perpendicularly intersecting straight lines (that would be the case, for example, with every azimuthal projection, for any two arbitrarily chosen perpendicularly intersecting H); rather, our classification term "double-symmetrical" presupposes that both symmetry-straight-lines be the base-line (representation of the basic-circle, Grundkreis) and the mid-H.

Here one should recall the earlier remark that various families can be ascribed to one map, according to which grid of primary and secondary circles of the globe are considered the basis of the representation. A transverse-axical gnomic projection, for example, (No. 1 of the system table) is considered, according to the generally accepted interpretation, a perspective projection with the eye-level at the middle of the globe, and such a projection represents the grid of the prime point as concentric full circles and their diameters. This map grid is centrally-circular and not double-symmetrical; according to the grid, the projection belongs in Family I. But if one wished to determine the law of representations of such a map according to the polar grid, then the projection would not be called centrally-circular, since its N are hyperbolae, but rather double-symmetrical, since the H and the N are arrayed symmetrically to the base-line and the mid-H. According to the polar grid, then, the map would have to be placed in Family III, the curved-symmetrical projection. Obviously, however, one will want to consider the far greater uniformity oi the perspective projection as definitive for systematic classification.

If, in the centrally-circular projections (Family I and II in general, Family IV and V in single instances), the mid-point S of the N is at the same time the representation of a primary-point of the globe [$r=0$ for $\delta=0$ and/or $x=\infty$ for $\phi=\pi/2$], then the map is a *primary point map.* If S is not the representation of a primary-point of the globe [$r>0$ for $\delta=0$ and/or x finite for $\phi=\pi/2$], then the map is a *ring-map.* On a ring-map, the representation of the primary-point of the globe on an N-line can be either a point (Punkt-Ringkarte or *point-ring-map*) or a line segment (*line-segment-ring-map*) or also a full circle (Kreis-Ringkarte or *circle-ring-map*).

With the projections which are *not centrally-circular* (Family III in general and most of Family IV and V), where the grid of the projection is in general always the polar grid, the map is called a *polar-point-map*, if at least one primary point of the globe is represented, and *polar-line-map*, if neither of the two primary-points of the globe is represented.

One might be reminded here of the ambiguity of the representation-equations. For example, the radius-law (Halbmessergesetz) $r=\delta$ (System no. 24) produces in Family I for the central-interval-true radial primary-point-map infinitely many different ring-maps for the inner-most map; and in Family II, infinitely many maps are produced, for example, for the flat-map (Plattkarte) (System no. 90) according to its equation $x=\phi$; $y=\lambda$, agreeing in the y-direction as well as in the x-direction, and congruent with the original map.

The Terms: secondary-circle-divided (nebenkreisteilig) and cellular (zenithal)

All the N are uniform in a *secondary-circle-divided* projection. For true-circular secondary-circle-divided projections, the mathematical expression in polar coordinates r and α is $\alpha= g(\delta) + \lambda\ h\ (\delta)$ and $r - F\ (\delta)$ with reference to conventional cylindrical in rectangular coordiantes $x = F\ (\phi)$ and $y = g\ (\delta) + \lambda\ h\ (\delta)$. If the N are curves, the more complex equations are necessary with respect to this property.

Cellular (zenithal) means centrally-circular with exchangeable, equal cells. Furthermore, with the property of being secondary-circle-divided, every H must be permutably equal to every other H. Such projections are especially valuable, because in them every H shows completely equal proportions. It is because of this radial symmetry that grids

of this type have such significance for nautical and astronomical conversion of coordinates[7]. The mathematical expression for the property of being cellular (Fächerigkeit, Zenitalität) is $\alpha = f(\delta) + n\lambda$; $r = F(\delta)$ and/or $y = f(\phi) + n\lambda$; $x = F(\phi)$, where n is a constant. The designation "zenithal" is very appropriate for this important branch of the system of map projections, since the zenithal line of the prime point of the globe is of great importance for every such projection. The great circles through them are represented as identical, permutable in every respect, with all secondary-circles around them as axis, as concentric circumference, which are at the same time equi-distorted (Verzerrungsgleichen, Äquideformaten). Every point of such a line possesses the same distortion-indicatrix, according to Tissot. Unfortunately, the word "zenithal" has been subject to great misuse.

One has to differentiate here the following arrangements in projections:

I. The N should be concentric circumferences around a center A. This is what we call *centrally-circular* (mittkreisig).

II. As I, but with the additional "namely full circles". This is what we call *radial-circular* (radkreisig).

III. As I, but with the additional "and at the same time equi-distorted." We call this *cellular* (facherig).

IV. The H are straight lines which intersect at a point A at the same angle as in the original on the globe. We call this *radial-directional* (radstrahlig).

V. = I and IV (radial-circular and radial-directional (radkreisig und radstrahlig) is called *radial* or *azimuthal*.

In our literature one finds the following definitions of "zenithal:"

1. Zenithal = oblique-axical in Herz 1885[8], Tissot-Hammer 1887[9], Hammer 1889[10].
2. Zenithal = the opposite of oblique-axical in Schoy 1913[11].
3. Zenithal = equal-cellular (gleichfächerig) in Zoppritz 1884[12], Hammer 1889[13], Maurer 1905[14], Zoppritz-Bludau 1912[15].
4. Zenithal = azimuthal in Gretschel[16].
5. Zenithal = centrally-circular in Hammer 1889[17].
6. True-zenithal = radial-circular in Hammer 1889[18].
7. Zenithal = true-shape in the mid-point of the map in Hammer 1915[19].

A few extracts from the cited literature in our bibliography illustrate this monstrous chaos, as discussed in more detail at other places by the author[20]. We use the expression according to the definition given above, as Zoppritz already used it in 1884, and for which I propose the germanization *cellular* (fächerig), a term surely not to be misunderstood. It is supposed to mean that all cells of the projection are congruent (kongruent). The N are equally-divided, equal-centered circumferences and the H are lines congruent with their sections (Einteilungen).

The Term: straight-cellular (geradfächerig) or general-conical (allgemein kegelig)

The subdivisions of the cellular projections, in which all H are straight lines, should be called straight-cellular (geradfächerig). We find straight-cellular projections in Family I, where they represent radial (proper conical) projections with unclosed arcs as N, as well as in Family II, in which the true-cylindrical projections are given. They are found even in Family IV among the less uniform projections, as no. 197, 198, and 224 of the System. All three are *conventional-cylindrical* as well as *cylindrical-radial*

(saulenstrahlig); but only the first two are cylindrical (saulig), as will be further illustrated on p. .

The mathematical expression for the straight-cellular projections if $z = F(\delta)$; $\alpha = n\lambda$ (n - constant) or, in the case of the cylindrical projections, $x = F(\phi)$; $y = n\lambda$. Even if the likenesses of the map in the border cases of the radial (n = 1) and of the cylindrical projections (n = 0) are very different from each other and from that of the conical projections (n neither 0 nor 1), the regularity of these three groups is so intercoupled that the compilation of the three classes is more or less emphasized as systematically expedient in most text books. The definition of the "conical projections" in Bougeois-Furtwängler[V] also encompasses the two border cases, but the designation "conical representations" is used without differentiation, regardless of whether or not the border cases are included. Herz[II], in his chapter on "Conical Projections," treats the definition of the azimuthal (he says zenithal) projections and goes into particulars about all the true-cylinder projections. Zöppritz-Bludau[VI] emphasizes the relationship on p. 30 and p. 104, as does Hermann Wagner[VII] on p. 204. In the literautre, one finds, next to the expression "conical in the more general sense," the designation "geometric simply defined," which stems from Hammer; one finds these terms for the compilation of the radial, conical, and cylindrical, that is, for out straight-cellular projections. The term does not designate anything which is essential in the matter, and it should be dropped once and for all. The globular projection (Globularprojektion) (no. 158 of the System), with its childishly simple geometrical definition, should not be included in the group of "geometric simply defined," but rather the Tissotian perigonal projections and the projections of James Clark, which are defined with complicated integral expressions. A better expression is *"general-conical"* (allgemeinkegelig), which makes absolutely clear that a proper-conical projection is meant, including the border cases, where the cone used degenerates to a plane surface or to a

cylinder. The shorter designation *straight-cellular* is also good; it is supposed to make clear that the congruent cells of the grid are bordered by straight prime lines.

The principle geometric concept which justifies the designation *conical projection* is this: that the two-dimensional map can be laid on the surface of a cone, which has the same axis as the globe, in such a manner, that the H of the grid appear as equi-distant cone-laterals and the N as circles of the cone. The half opening-angle (Offnungswinkel) γ of the cone is determined by the constant n of the cone-projection in the designation sin γ=n, since, with a cone of half opening-angle γ whch results from turning the straight lines SK (cf. illustration 1, table 1) around the straight line SO, the angle drawn of the lateral of the cone SK onto the shell is $\alpha=n\ \lambda=\lambda \sin\gamma$, if the rotating plane surface SOK describes the angle λ; for, $\sin\gamma$=OK : SK=n. Since sin γ cannot be greater than 1, there is a conflict in accepting the principle geometric concept n>1. Nevertheless, one could project maps according to the laws $r=f(\delta)$; $\alpha=\lambda$; n>1. But there is, however, hardly any intelligent reason for making such a choice of n. On the other hand, one must maintain a clear understanding that with n<1, that is, with a real conical projection, spreading the cone onto a plane surface leads to numerous repetitions of the representation of the world map. If n is a rational number, then a finite number of variously oriented world map representations result, which join each other without overlapping when n=1/2 and m=whole number. Here the illustration of a closed curve of the globe is also a closed curve on the representation.

The distance of the tip of the cone S from mid-point O of the uniform-globe can be chosen in various manners. If it [the distance] is = 1/n (S moves toward S' in illustration 1), then the cone is contiguous with the globe at secondary circle $\delta' = \pi/2 - \gamma$ = arc cos n; [δ' =∢ S'OD']. If SO<1/n the cone intersects theglobe at two secondary circles δ_1 and δ_2;

[$\delta_1 = \sphericalangle\ SOD_1$; $\delta_2 = \sphericalangle\ SOD_2$]. Cases $SO > 1/n$, in other words, where the basis is a cone which freely encircles the globe, are reserved in our system for the exceptional case, in which the cone degenerates to a cylinder (no. 86 of the System).

SECTION II: THE SYSTEM TABLE

(table at the end of this volume)

Prefatory Remark

Now that we have clarified the general terms, we begin with the system table. The families of the system and their branches are given in single headings and the further subdivisions (subbranches, orders, classes, etc.) are listed in vertical columns, while the lines indicated with running "system numbers" (Abbreviation: SNo.) in the first column give the particular projection types according to their distinctive characteristics. The column labeled "Field Number" (abbreviation FNo.) refers to the table of terms in map projection, as given and commented upon in Section I. In the wide column labeled "Type", one finds particulars concerning name, time of origin, equations and other particulars about the projection, while the final column refers to the literature about it. The original source is mentioned only in isolated instances, whereas the well known text books are often mentioned, using the abbreviations for them found at the beginning of our bibliography. The numbers after these abbreviations always mean page numbers; only in the case of no. 111, <u>Tissot-Hammer</u>, *are paragraphs of Part II referred to.*

INTERPRETATION OF THE SYSTEM TABLE

Family I, Branch A: true-circular and secondary-circle-divided (SNo. 1 - 73)

Subbranch A: Cellular; Order a: Straight-cellular (SNo. 1-52)

Class I: Radial (azimuthal) (SNo. 1-31)

Among the straight-cellular projections we find the border case n=1 of the *radial* (azimuthal) projections, which are defined mathematically by the equation $\alpha=\lambda$; $r=f(\delta)$. The property $\alpha=\lambda$, i.e., that the H are straight lines which intersect each other at one point with the same angle as on their original, is called *radial-directionality* (Radstrahligkeit), while the property that the N are concentric full circles is called *radial-circularity* (Radkreisigkeit). The definition *radial (azimuthal) = radial-directional + radial-circular* in which the point of intersection of the radial-rays is at the same time middle-point of the radial circle, is common to all of cartography. The deviating definitions of Schoy[11] that azimuthal = radial-directional + radial-circular if one is to expect to call the illustration a map at all, that azimuthal = radial-directional with or without radial-circularity, if the illustration is supposed to be a cartogram - have been justifiedly rejected as unfortunate[20], since scientific definitions must obviously be unequivocal. Hammer's demand, that an illustration only be called a map if it represents a particular part of the surface of the earth with the least possible distortion, is much too broad. According to his stipulations, maps of the moon and the straight-directional (geradwegig, i.e. orthodromic) maps are only cartograms; but obviously the purpose of an illustration has nothing to do with the terminological rule about what is azimuthal.

Group A: plane-perspective (perspective) (SNo. 1-21)

In the class of radial maps we must differentiate between prime-point maps and ring maps, as we have just made clear above with respect to straight-cellular maps. Among the radial prime-point maps, all of which are true-shape at the prime-point of the map, there is one group which is particularly prominent, the group of perspective projections, which give a perspective picture of the surface of the globe. (The cartographer, too, may use the word perspective - perspektiv -

for perspectival - perspektivish -, as has been the practice in geometry for more than half a century.) The persepctive illustration is projected from a point of vision A either onto a plane surface, which stands perpendicular to the straight lines AO [O = middle-point of the uniform globe], or onto a cone or cylinder, which have the straight line AO as axis. Accordingly, one differentiates between plane-perspective (ebensichtig), cone-perspective (kegelsichtig), and cylinder-perspective (saulensichtig) projections. Only the plane-perspective projections are of greater significance, so that one usually understands perspective as meaning plane-perspective.

The particular *plane-perspective* projections (illustration 2, table 1) differ in distance q = OA between point of vision A and globe-center O. The distance c = OM between globe-center and the surface of the plane representation changes only the scale on the map; it has a significance with respect to the type of projection only if exact agreement of measure of surfaces and distance-intervals between map and uniform-globe is demanded; this is sometimes overlooked. (Examples of the latter in SNo. 10-19).

Illustration 2 shows the radius-law as law for all plane-perspective projections:

$r = (q + c) \sin \delta : (q + \cos \delta)$ and the stipulation pertains.

We can differentiate as types according to orientation of the point of vision:

Internal-perspective (innensicht[illegible]) with point of vision in the globe [$-1<q<1$].

External-perspective (außensichtig) with point of vision outside the globe [$q>1$], between which stands the *globe-surface-perspective* projection, with point of vision on the globe jacket [$q=1$].

Since one line of vision strikes the globe generally at two points, every two points of the globe are reproduced as one point of the two-dimensional representation, when any perspective projection of the whole globe is involved. The secondary-circle $\delta=\Delta'$ is given in the system table for the external-perspective projections [$\Delta = \measuredangle$ NOD' in illustration 2], the representation-circumference of which is the outer border of the doubly accomplished world map. One usually constructs plane-perspective maps only a picture of a dome of the globe, from $\delta=0$ to $\delta=\Delta'$ in the case of external-perspective, and in the case of internal perspective, from $\delta=0$ to δ', where $\cos \delta' = -q$, becoming thus $r=\infty$. The arrangement of representation is not so as to correspond to the lines of vision which reach the dome of the globe from inside the globe, but, rather, to correspond to the lines of vision which gravitate outward toward the point of vision. The plane-perspective map is thus really the perspective picture of the surface of the earth as seen from a point of vision lying above it, only in the one instance when $q=\infty$, point of vision infinitely distant.

Of the internal-perspective projections there is only one of value, the *center-perspective* ($q=0$) SNo. 1, where the point of vision lies at the center of the globe. It is usually called a *gnomonic* projection or central-perspective and has a most valuable property, that of representing all geodetic lines of the globe, the great-circles of the globe, as geodetic lines on the plane, i.e., as straight lines. Representations which accomplish thi are called *orthodromic* or *straight-directional*. The center-perspective projection presents also a special example of the *true-great-circle* projections which represent all global-great-circles as circumferences, if, of course, those circumferences become straight lines. Schols derived the radius law $r=2 \operatorname{tg} \delta : (1 + \sqrt{1 + k^2 \operatorname{tg}^2 \delta})$ [k=constant, in particular k=0 for the center-perspective projection]. There are also *non-radial*

true-great-circle projections, since the great circles remain straight lines on every collinear illustration of a gnomonic map; the map thus remains true-great-circle as well as straight-directional, while thereby losing not only the property of being radial (azimuthal) but also the property of being centrally circular (axial). One should, therefore, not call such projections gnomonic, as has often been done in the literature[22], since this name befits only the axial and only the central-perspective with true shape at the prime points, while affine representations of these maps are generally true-shape at two points and have in this manner achieved a certain amount of practical significance (cf. SNo. 182 and the bibliographical note about this on p.118). In a center-perspective projection all the meridians are always straight lines, no matter how the prime grid lies, but the parallel circles are not always hyperbolae, as Zoppritz-Bludau[23] proposes, but rather, they can be parabolae, ellipses and circles. In such a projection the map of one hemisphere alone fills up the whole surface of the picture. The suggestion in Tissot-Hammer[24], that the significance of the Mercator projection (SNo. 94.95) is retreating more and more in favor of the gnomonic marine map, has hardly been supported up to now; except for a few specialized maps, the marine maps have remained Mercator maps.

The second type of the perspective projections (SNo. 2) is the *globe-surface-perspective* (q=1) kind, which was first used by Hipparch and since 1613 has generally been called *stereographic*, or body-delineating (which says nothing at all). The name does not designate any of the properties of this type of projection which is true-shape and reproduces all circles of the globe as circles. Since, among all the types of projection, this kind alone has this property, it would be fitting to find a German name for it: *all-circular* (allkreisig). Naturally, this projection is great-circle-true. Its radius law stems from Schols' formula for the

case k = 1; it is $r = 2 \text{ tg } \delta/2$. In the case of an all-circular map, the picture of the complete globe just fills out the whole plane surface of the representation.

The *external-perspective* projections SNo. 3-6, as for No. 1 and 2, the distance c of the plane surface of the picture from the center of the globe can be chosen arbitrarily. SNo. 3, the *orthographic* or *distant-perspective* (fernsichtig) map made by Apollonius in 240 B.C., reproduces the correct perspective picture of the globe from a huge distance; for this reason it is used for maps of the moon and the planets. It could also be used for certain geophysical maps and for antipodal maps, since, on the one hand, it reproduces true to scale the straight-line interval of every point of the earth from a diameter of the earth, and, on the other hand, reproduces each two antipode points as end points of the diameter of concentric circles.

The claim of the projection (SNo. 4) by De La Hire $r(90°) = 2r\ (45°)$ could pertain, not just for the hemisphere, but also for a calotte Δ in the form of $r(\Delta) = 2r(\Delta/2)$. I find, instead, (SNo. 5) the designation $r = 1 + \cos\ \Delta/2$. Projection (SNo. 6) is mentioned in Gretschel and Herz without reference to the name of the originator.

With SNo. 7-19, c is no longer arbitrary. SNo. 9, the projection with the minimum of greatest distortion of length at value c=1, is found in Gretschel, while Parent (No. 8) mentions only the case c=0.

Projections 10-15, with respect to hemispheres, and 16-21, with respect to globe domes with arc radius Δ, call for such stipulations regarding scale that c can no longer be chosen arbitrarily, when q is given for the uniform globe. For No. 10-13, Parent has mentioned only the cases in which c=0, while in Gretschel and Herz the values are found from q for c=1.

The following should be mentioned about the projection No. 16 by Fischer: According to Tissot-Hammer, No. 63, the characteristic stipulation $r(\Delta) = \Delta$ is treated for this projection with the unfounded assumption that $c = 1$, and accordingly the equation

$$q = (\sin \Delta - \Delta \cos \Delta):(\Delta - \sin \Delta)$$

is derived, resulting in $\Delta = 90°$ $q = 1.752$. As a matter of fact, c can also have other values, so that the equation should be $q = (c \sin \Delta - \Delta \cos \Delta):(\Delta - \sin \Delta)$. Thus, for a specific value Δ, one can fulfill Fischer's stipulation $r(\Delta) = \Delta$ by means of infinitely many value pairs (q/c). In order to obtain particular value pairs (q,c) one can require, for example, the stipulation $r(\Delta) = \Delta$ for two globe domes Δ_1 and Δ_2 Examples of this are given under No. 17.

One finds the same error in Hammer's projection (No. 18) as is Tissot-Hammer. Again, $c = 1$ is unjustifiedly stipulated. In place of the equation $(q + 1) \cos \Delta/2 = q + \cos \Delta$ the equation $(q + c) \cos \Delta/2 = q + \cos \Delta$ should be stipulated. Here, too, one can expect equal-area for the two globe domes Δ_1 and Δ_2. Now one gets two equations for q and c: $q = 1 + 2\cos \Delta_1/2 \cos \Delta_2/2$; $c = 2 (\cos \Delta_1/2 + \cos \Delta_2/2) - q$. An example of this is given under No. 19.

Projections No. 12, 13 and 21 as to surface areas, and 8 - 11, 14, 15 and 20 as to lengths, claim the least distortion ratios; 8 and 9 and/or 14 are from time to time special cases of 20 and/or 15. Tissot-Hammer note inappropriately that projection 10, with least possible distortion on a hemisphere, has never been proposed; this projection is also mentioned in Gretschel and Herz as a projection of Parent stemming from the year 1702.

Non-perspective radial projections (SNo. 22-31)

The latter two of the non-perspective radial prime-point maps (No. 22-27) also make certain claims of least distortion. But one must observe that Airy's projection is not the exception for the hemisphere of James-Clark's projection, as one might assume from the description on p. 221 in Herz; rather, the integral stipulations of both projections refer to different basic dimensions. Assuming that for every point of the map k_1 and k_2 are the greatest and the least ratios of length distortion, James Clarke (No. 27) presents both expressions $(k_1 - 1)^2$ and $(k_2 - 1)^2$ as scale dimensions for length distortion, while Airy (No. 26) gives $(k_1/k_2 - 1)^2$ also as scale dimension for length distortion, but $(k_1k_2 - 1)^2$ for area distortion. The formula given for r in the system table corresponds with the description by Herz (p. 221) who avoids the inappropriate stipulation with regard to constants, to which Clarke and James, in agreement with Airy, and even Gretschel (p. 98), yet adhere.

The Lidman map (No. 22) is a special case of a universal doubly-projected projection. If one first projects the globe onto a coaxial cone, and then from a point of vision lying on the axis of projection on a flat surface perpendicular to this axis, then the radius law of the thereby resulting radial point-map is:

$$r = \frac{\sin\delta\ (q_2 + c)\ (h + q_1)\ \mathrm{tg}\ \gamma}{\mathrm{tg}\ \gamma(q_\gamma + \cos\delta)\ (h + q_2) + (q_2 - q_1)\ \sin\delta}$$

In this equation q_1 and q_2 are the intervals of the first and the second points of vision from the center of the globe; h and c are the intervals of the tip of the cone, and/or of the picture surface, measured in opposite direction from the center of the globe, while γ represents the half opening-angle of the cone. Since five constants q_1, q_2, h, c, γ

are at our disposal here, one could fulfill many different demands with such a projection. Lidman's exception corresponds with the stipulations $q_1 = c = 0$; $q_2 = \infty$; $h = \sqrt{2}$; $\gamma = 45°$.

The expression "intervening" (vermittelnd), as applied to projections 24-27 originated with Hammer (in place of the expression "true-center" (mitteltreu) chosen by von Breusing for his projection) and it is supposed to mean that the dimensions of distortion of these projections are just about the mean between the dimensions for distortion for equal-area and for true-shape projections.

Radial ring-maps (No. 28-31) have little practical use, except sometimes for the equal-area No. 28, which no doubt shows more distortion of length of all N, as well as distortion of shape, than the equal-area point-map No. 23. Ring-maps No. 29 (equal-area) and No. 30 (circle-interval-true), which one may consider a plane intersecting at the scale-true secondary circle δ_m of the globe, could at least be sensibly used for representations of a zone of the globe in the vicinity of these N. If one draws a true-shape map with scale-true secondary circle δ_m on the above mentioned plane, then it becomes geometrically similar to the all-circular projection No. 2, except for the change in the scale, by which the radius law now reads $r = (1 + \cos \delta_m) \operatorname{tg} \delta/2$ rather than $r = 2 \operatorname{tg} \delta/2$. It becomes, then, as all true-shape maps, a point-map.

Family II: Conical (SNo. 32-52)

A cone which is coaxial with the uniform globe is the basis of the conical projections; it may not degenerate to a plane or to a cylinder. In the law of prime-lines $\alpha = n\lambda$, which means the property of being *conical-radial*

(Kegelstrahligkeit), n has to be a proper fraction, namely $n = \sin \lambda$, where λ is the half opening-angle. Even if it doesn't matter how far one imagines the tip of the cone S (Illustration 1, table 1) to be from the center of the globe O, it is usually useful to assume a certain interval OS = c, whereby the cone can become a *contiguous cone* (S inclines toward S') or an *intersecting cone* (SO<S'O). In the case of the contiguous cone $n = \sin \gamma = 1/c = \cos \delta_m$ [$\delta_m = \measuredangle$ S'OD' where the side lines of the cone S'D' are contiguous with the globe]. The intersecting cone intersects the globe in both secondary-circles δ_1 and δ_2. Here $\delta_1 = 90 - \gamma + \varepsilon = \measuredangle SOD_1$; $\delta_2 = 90 - \gamma - \varepsilon = \measuredangle SOD_2$; and $n = \sin \gamma = 1/c \cos \varepsilon$, where $\varepsilon = \measuredangle D_1OE = 1/2 \measuredangle D_1OD_2$.

A *contiguous-cone-map*, using representation law $\alpha = \lambda \sin \gamma$, is considered as originating on the cone with the half opening-angle γ which is contiguous with the uniform globe at the scale-true reproduced secondary-circle $\delta_m = 90° - \gamma$. The radius law, then, suffices for the stipulation $r(\delta_m) = \cot \gamma$.

An *intersecting-cone-map*, using representation law $\alpha = \lambda \sin \gamma$, is considered as originating on the cone with the half opening-angle γ, which intersects the uniform globe at a scale-true reproduced secondary-circle δ_m, where $\delta_m \gtrless 90° - \gamma$. Then the cone intersects the globe at yet another secondary-circle [$180° - 2\gamma - \delta_m$], which generally is not reproduced true to scale.

Beside the property of being conical-radial, there is also the second property of the conical projections, the property of being *conical-circular* (Kegelkreisigkeit), which is accomplished with the formula $r = f(\delta)$. Thus, *conical = conical-circular + conical-radial*, and the point of intersection of the rays of the cone must be at the same time the middle-point of the cone-circles.

Conical Prime-point Maps (SNo. 32-44)

Here we are differentiating between the subclasses of prime-point maps [$r_0 = f(0) = 0$] and of ring-maps [$r_0 = f(0) > 0$]. Among the former appears first a group of *cone-perspective* projections, that is, perspectives of the globe on a coaxial cone as seen from point of vision A on the axis of the cone. If the tip of the cone S (Illustration 1) lies about c = OS above, and the point of vision A about the straight line segment q = OA below the plane of the basic circle GOB, then the radius law for the general case of the *cone-perspective* projection results (SNo. 32)

$r = (q + c) \sin\delta : [q \sin \gamma + \sin (\gamma + \delta)]$, which in the case of $c = \operatorname{cosec} \gamma$ is valid for contiguous cones (No. 35). The law from No. 32 is carried over for $\gamma = 90°$ to the law for the plane-perspective projections (p. 24). The law for the cylinder-perspective projection (SNo. 92), which is given later (p.61) is derived a $y = r - c$, when one lets $\gamma = 0$, $c = \infty$ and $c \sin \gamma = \cos$ becomes 0_m, where 0_m are scale-true N.

SNo. 33 gives the exception to the *middle-perspective conical projections* (q = 0). In Tissot-Hammer it is called "laying out the gnomonic projection onto a cone." It can also be contiguous-conical (SNo. 36).

The *globe-surface-perspective conical projection* (q = 1), with the point of vision on the surface of the globe, has the general equation $r = (c + 1) \sin \delta: [\sin \gamma + \sin (\gamma + \delta)]$. In the table, only the special case of a contiguous cone $\delta_m = 60°$ is given (SNo. 37), which has been named Brauns stereographic conical projection.

SNo. 34 gives Murdochs *middle-perspective equal-zone conical* projection (known as Murdoch II), in which the globe zones between the N δ_1 and δ_2 are *not* represented *true-area*,

but are on the whole *equal-area*. With such a stipulation the cone must have a very particular opening and inclination toward the uniform globe, for which the equation

$$\gamma = 90° - \frac{\delta_1 + \delta_2}{2}; \quad c = \sec\frac{\delta_1 + \delta_2}{2}\sqrt{\cos\frac{\delta_1 - \delta_2}{2}}$$

is applicable. Here neither δ_1 nor δ_2 are the scale-true N δ_m through which the cone extends. For them the equation $\sin\delta = r\sin\gamma$, where for r the radius law is

$$r = \sqrt{\cos\frac{\delta_1 - \delta_2}{2}}\left[\operatorname{tg}\frac{\delta_1 + \delta_2}{2} + \operatorname{tg}\left(\delta - \frac{\delta_1 + \delta_2}{2}\right)\right]$$

For $\delta_1 = 60°$, $\delta_2 = 30°$, for example, one finds c - 1.39, and, for the scale-true δ_m the values $\delta_m = 55°38.3'$ and $\delta_m = 34°21.7'$.

Conical projections, too, can be zone-perspective, where every secondary circle is projected from its middle-point onto the cone. In T.-H.[25] the zone-perspective conical projections are quite unfortunately called "cone lay-offs" (Kegelabwicklungen), a designation which could, of course, be applied to any arbitrarily chosen conical projection. The general equation for such projections is $r = (c - \cos\delta) : \cos\gamma$. Such a map can be a prime-point map only in the case $c = 1$, where the tip of the cone inclines to the surface of the globe. A projection of this type is given under SNo. 38. $\delta_m = 180° - 2\gamma$ is valid for the true to scale N.

The *true-shape* conical projections (group C) are, as are all true-shape maps, point maps. They correspond in both forms No. 39 and 40 with the projection of Lambert-Gauss. No 39 (contiguous-conical) represents only one N, and No. 40 (intersecting-conical) two N true to scale. With equal value $n = \sin\gamma$, both projections are geometrically similar, but they cannot be called identical - as is done in Z.-B.[26], since

they are not identical with respect to the uniform globe. In No. 39 (contiguous-cone) all N, except for one scale-true $\delta_m = 90° - \gamma$, are enlarged in the reproduction. In No. 40 the intersecting-cone used for the reproducing is not the one which passes through the two scale-true secondary-circles δ_1 and δ_2; for this cone would have a $\gamma = 90° - \frac{\delta_1 + \delta_2}{2}$, while in the case of a true-shape projection with true to scale N δ_1 and δ_2 it would have to be

$$n = \sin\gamma = \frac{\log\sin\delta_1 - \log\sin\delta_2}{\log tg\,\frac{\delta_1}{2} - \log tg\,\frac{\delta_2}{2}}$$

These examples $\delta_1 = 90°$; $\delta_2 = 30°$ will serve for the elucidation of the ratios. Then $n = \sin\gamma = 0.5263$; $\gamma = 31°45.4'$ results. When reproducing onto a contiguous cone of such an opening and which is contiguous with the uniform globe at the secondary-circle $\delta_0 = 90° \cdot \gamma = 58°14.6'$, where only this N is true to scale, while all other N appear enlarged, the radius law is $r = 2.198\left(tg\,\frac{\delta}{2}\right)^n$. The tip of the cone lies, then, at an interval $c = \sec\gamma = 1.900$ above the basic circle of the uniform globe. But in the case of reproductions, which give both secondary-circles $\delta_1 = 90°$ and $\delta_2 = 30°$ true to scale, one can choose either the intersecting-cone I through the secondary-circle $\delta_1 = 90°$, which produces the cone-tip-interval $c = 1.616$, or the intersecting-cone II through the secondary circle $\delta_2 = 30°$ with a cone-tip-interval $c = 1.674$. In both cases, I and II the radius law is $r = 1.900\left(tg\,\frac{\delta}{2}\right)^n$; in both instances both N $\delta_1 = 90°$ and $\delta_2 = 30°$ are reproduced true to scale and the secondary circle δ_0 to the greatest extent, namely at a ratio of 1.900: 2.198. The second intersecting-circles of both cones are, however, not reproduced true to scale, in the case of cone I $\delta = 26°29.2'$, in the case of cone II $\delta = 86°29.2'$; the first is enlarged, the second made smaller.

Since the $n\lambda$, rather than the angle λ appear on the conical prime-point maps between the straight-line H at the prime-point

of the map, it is often stated in the literature, for example in T.-H.[27], that in true-shape conical projections the prime-point of the map is a *singular point at which the property of true-shape, which exists everywhere else, is missing.* This is a mathematical error, despite deceiving appearance. The picture of a full great circle of the globe is, of course, not a straight line which passes through the prime-point of the amp, but, rather, it consists of two straight rays which enclose the angle $n\pi$. At this point, the curve which consists of two half straight lines has a whole bundle of tangents; and exact mathematical inquiry, as to which of all these tangents is applicable in a single instance, shows that the property of true-shape is kept at this point, too. Even in the case of the infinitely distant straight line of the surface of the map, the representation of the other prime-point of the globe conforms to the continuing existence of the property of true-shape, as one can prove by means of reproducing with reciprocal-value rays (transformation with reciprocal radials). The same applies, also, when the true-shape map changes to a cylindrical map; thus, this applies for the Mercator maps (SNo. 94), the parallel meridians of which intersect each other in infinity at the proper angles despite everything. In a more general sense, the true-shape radial, conical or cylindrical projections (Sno. 2, 39, 40, 94, 95) are special cases of Lambert-Lagrange's true-shape circle-grids (SNo. 169-173); here also the property of true-shape remains constant at every singly point, if one makes exact mathematic inquiry, whereas the deceiving appearance can cause one to miss the fact. Maurer[28] published the evidence of this in 1919.

In the case of a *central-interval-true conical prime-point map*, according to the equations $\alpha = n$; $r = \delta$ (SNo. 41), every N is true to scale for which the $n\delta_m = \sin\delta_m$, and that dome of the globe (arc radius Δ) is reproduced equal-area, for which $\Delta\sqrt{n} = 2\sin\Delta/2$. It is interesting to note that in this special case (No. 42) one can fulfill both stipulations for the same value $\delta_m = \Delta$. This succeeds for $\delta_m = \Delta = 74°52'$, if

one chooses $n = \sin\gamma = 0.7388$, thus $n \cdot 360° = 266°$; $\gamma = 47°38'$. The tip of the intersecting-cone through the secondary circle $\delta = 74°52'$ has, then, the interval $c = 1.1417$ above the basic-circle plane of the globe.

The first projection of Schjerning (No. 43) is also a *central-interval-true conical prime-point map*. It originates from the radial central-interval-true prime-point map (SNo. 21), in which the angles α are changed in ratio n, the rays r, however, remaining = δ. Z.-B.[29] declares completely incorrectly that this is not a conical projection. Bludau gives the example $n = 1/2$ (P. 189, Fig. 119). This is naturally a proper conical projection on the intersecting-cone ($\gamma = 30°$) through the scale-true reproduced secondary circle $\delta_m = 108°36'$, since n is $\delta_m = \sin\delta_m$ for $n = 1/2$. On this conical map there is *equal-area* (not true-area!) for the dome of the globe $\Delta = 159°27'$, since $\Delta\sqrt{1/2} = 2\sin\Delta/2$ applies for this value .

A *true-area conical prime-point map* has, according to Lambert (No. 44), the equation

$$\alpha = n\lambda;\ r = 2\sin\frac{\delta}{2} : \sqrt{n}$$

It corresponds to an intersecting-cone through the scale-true reproduced secondary circle δ_m, where $\cos^2\frac{\delta_m}{2} = n$, and it shows equally great distortion of length and shape at the N δ_1 and δ_2, where $n = \cos\frac{\delta_1}{2}\ \cos\frac{\delta_2}{2}$. Germain christened this projection of Lambert an "isopheric stenoteric projection." Gretschel gives its grid as $\delta_1 = \delta_2 = 45°$ in table IV, fig. XX. In order to achieve the least possible distortion of uppermost angles at the dome of the globe between $\delta_2 = 0$ and δ_1, one must choose $n = \cos\frac{\delta_1}{2}$ and then lay the intersecting-cone through the thereby scale-true reproduced secondary circle δ , where soc $\cos^2\frac{\delta_m}{2} = \cos\frac{\delta_1}{2}$. For a world map ($\delta_2 = 0$; $\delta_1 = 180°$) this stipulation would result in the least possible uppermost angle

distortion $n = 0$. The auxiliary cone and the world map then shrink together into a straight line. This unhappy result is avoided in T.-H.[30] by saying that for a *world map* one must require equally great distortion, not at the north and south poles, but, rather, at the north pole and at 50° south latitude. For this the peculiar reason is given that the distortion ratios in the southern latitudes do not matter. The thus recommended area-true map of the world fills a sector of only 123.1°.

Conical Ring Maps (SNo. 45-52)

In the following subclass of *conical ring maps*, which represent the global point $\delta = 0$ as an arc with the radius $r_0 > 0$, we first find the group of *zone-perspective* maps. As intersecting-maps they have two scale-true N δ_1 and δ_2, from which the stipulations for the projection (SNo. 45) are drawn;

$$n = \cos\frac{\delta_1 + \delta_2}{2}; \quad c = \cos\frac{\delta_1 - \delta_2}{2} \sec\frac{\delta_1 + \delta_2}{2};$$

$$r = (c - \cos\delta)\operatorname{cosec}\frac{\delta_1 + \delta_2}{2}; \quad \gamma = 90° - \frac{\delta_1 + \delta_2}{2}$$

With $\delta_2 = 0$ they pass into the zone-perspective prime-point maps (No. 38); with $\delta_2 = \delta_1$ we get zone=perspective contiguous-cone projections (No. 42) with only one scale-true N δ_1 and the equations $n = \cos\delta_1$; $c = \sec\delta_1$; $r = (\sec\delta_1 - \cos\delta_1)\operatorname{cosec}\delta_1$; $\gamma = 90° - \delta_1$. No. 46, with the deterioration of the contiguous-cone into a cylinder (in the case $\delta_1 = 90°$), becomes the area-true cylindrical projection of Lambert (SNo. 87).

In the second group, that of *circle-interval-true* map with only one scale-true N δ_m, we find the oldest conical projection, the so-called simple conical projection of Ptolemaus - from the second century AD - on the contiguous-cone of the

scale-true reproduced secondary-circle (No. 47). The second circle-interval-true projection, projection No. 48, with the two scale-true N δ_1 and δ_2 stems from De l'Isle (1745); Z.-B.[32] shows that this projection was falsely attributed - e.g., in T.-H.[31]- to Mercator. Both of these projections, No. 47 and No. 48, should not be caller *center*-interval-true, since there is no middle point on the map from which the intervals can be accurately reproduced, because the illustration of that point of the globe, from which all intervals are measured true to scale, is an arc. The auxiliary cone of the projectio of De l'Isle cannot pass through the scale-true reproduced secondary circles δ_1 and δ_2, since this intersecting-cone does not result in the values appropriate to the projection:

$$n = \sin\gamma = \cos\frac{\delta_1 + \delta_2}{2}\sin\frac{\delta_1 - \delta_2}{2} : \frac{\delta_1 - \delta_2}{2}$$

However, one can lay the auxiliary-cone as an intersecting-cone through one or the other of the secondary-circles δ_1 and δ_2. One can also lay the auxiliary-cone as a contiguous-cone on the secondary-circle δ_3 which exhibits the greatest shape and area distortion in the zone between δ_1 and δ_2, where $\cos\delta_3 = n$. In the example in the system table ($\delta_1 = 70°$; $\delta_2 = 20°$) becomes $\delta_3 = 46°46'$; and the cone tip interval c_1, in the case of an intersecting-cone through circle δ_1 becomes $c_1 = 1.341$, the interval in the case of an intersecting-cone through circle δ_2 becomes $c_2 = 1.303$, and the interval in the case of a contiguous-cone through circle δ_2 is $c_3 = 1.460$. When $\delta_2 = 0$ projection No. 48 comes into the same classification as the center-interval-true prime-point map No. 41.

The conical-projection Murdoch I (1759), SNo. 49, is also circle-interval-true; it reproduces the zone of the globe between δ_1 and δ_2 equal-area on the whole (*not area-true!*) on an intersecting-cone, the γ and c of which result from the further stipulations, that the secondary-circle $\frac{\delta_1 + \delta_2}{2}$ should be projected from the center of the globe through the rays

perpendicular to the surface of the cone. The result is

$$n = \sin\gamma = \cos\frac{\delta_1+\delta_2}{2};\ r = \left[\sin\frac{\delta_1-\delta_2}{2}\ \mathrm{tg}\frac{\delta_1+\delta_2}{2} : \frac{\delta_1+\delta_2}{2}\right] - \frac{\delta_1+\delta_2}{2} + \delta$$

$$c = \sec\frac{\delta_1+\delta_2}{2}\ \sin\frac{\delta_1-\delta_2}{2} : \frac{\delta_1-\delta_2}{2}$$

This map can never be a point-map. It shows the secondary-circle $\delta = \delta_m$ true to scale, when $n \cdot r(\delta) = \sin\delta$. Example:

$\delta_1 = 90°$; $\delta_2 = 0°$ yields $\gamma = 45°$; $n = 1:\sqrt{2}$; $n \cdot 360° = 255.5°$;
$c - 1.735$; $r\ (\delta = 0) = 0.4414$; $\delta_m = 16.92$

In the case of the conical projection Murdoch III (1759(SNo. 50, which, according to T.-H.[33], is likewise circle-interval-true, the only difference from Murdoch I stated there is that the auxiliary-cone has been displaced parallel to itself, until

$$c = \left(\sec\frac{\delta_1+\delta_2}{2}\ \cot\frac{\delta_1-\delta_2}{2}\right)\frac{\delta_1-\delta_2}{2}$$

Since, as for Murdoch I, $n = \cos\frac{\delta_1+\delta_2}{2}$ also for Murdoch III, one derives for Murdoch III

$$r\left(\delta = \frac{\delta_1+\delta_2}{2}\right) = \left(\mathrm{tg}\ \frac{\delta_1+\delta_2}{2}\ \cot\frac{\delta_1-\delta_2}{2}\right)\frac{\delta_1-\delta_2}{2}$$

Thus, since $r\left(\delta = \frac{\delta_1+\delta_2}{2}\right)$ is greater for Murdoch III than for Murdoch I, the representation of the zone between δ_1 and δ_2 on Murdoch III would be greater than its original on the globe, if one, as with Murdoch I, makes the H true to scale, that is, makes the projection circle-interval-true. One must rather not make Murdoch III circle-interval-true, as is correctly stated in Gretschel[34], but, rather, distribute the H between the givens radii at δ_1 and δ_2 uniformly. The H are not interval-true, but are only interval-similar with the factor of similarity $m = \left[\sin\frac{\delta_2-\delta_1}{2}\ \mathrm{tg}\ \frac{\delta_2-\delta_1}{2}\right] : \left(\frac{\delta_2-\delta_1}{2}\right)^2$; the factor cannot become =1 as long as δ_1 and δ_2 are different, that is, as long as there is a zone which is reproduced equal-area. Naturally

one could call the projection Murdoch III - even with respect to another uniform globe - interval-true rather than interval-similar; but then the area-equality of the zone between δ_1 and δ_2, which is the reason for which Murdoch thought up his three projections, would not exist. The N $\delta = \delta_m$ also become true to scale where $n \cdot r\ (\delta = \delta_m) = \sin \delta_m$.

In the group of the *area-true conical ring-maps* we first find a contiguous-conical projection (SNo. 51) *with only one scale-true* N δ_m. This equation applies for the projection:

$$a = n\lambda;\ n = \sin \gamma = \cos \delta_m;\quad r_0 = \sec \delta_m - 1;$$

$$r = \sqrt{r_0{}^2 + \frac{4}{n} \sin^2 \frac{\delta}{2}} = \sqrt{1 + \sec^2 \delta_m - 2 \sec \delta_m \cos \delta}$$

All other N appear enlarged. The map can become a point-map on a proper cone, since $r_0 = 0$ $\sec \delta_m = 1$, that is, the cone becomes a plane. The two N, δ_1 and δ_2, where $\cos \delta_1 + \cos \delta_2 = 2 \cos \delta_m$, undergo equal enlargement respectively.

If an area-true conical ring-map is expected to reproduce *two secondary-circles*, δ_1 and δ_2 , true to scale, then n and r_0 must be stipulated according to the equations $n\ r(\delta_1) = \sin \delta_1$ and $n\ r(\delta_2) = \sin \delta_1$ where generally

$$r^2 = r_0{}^2 + \frac{4}{n} \sin^2 \frac{\delta_2}{2}\ .\quad \text{This results in}$$

$$n = \cos \frac{\delta_1 - \delta_2}{2} \cos \frac{\delta_1 + \delta_2}{2} \quad \text{and } r_0 = \frac{2}{n} \sin \frac{\delta_1}{2} \sin \frac{\delta_2}{2}$$

as equation for the projection of Albers (No. 52). When $\delta_2 = \delta_1$ No. 52 passes into the classification of No. 51; when $\delta_2 = 0$ into that of the prime-point map No. 44. No. 44 and 50 are thus special cases of No. 52, which represent the general form of all area-true conical projections. All N between the scale-true δ_1 and δ_2 are made less, all the others greater. The N δ_3 undergoes the strongest reduction, where

$\cos \delta_3 = n - \cos \frac{\delta_1 + \delta_2}{2} \cos \frac{\delta_1 - \delta_2}{2} = \sin \gamma$. The auxiliary-cone which is the basis for this illustration is not the one that passes through the scale-true reproduced secondary-circles δ_1 and δ_2, the n of which would equal $\sin \gamma = \cos \frac{\delta_1 + \delta_2}{2}$, whereas our $n = \cos \frac{\delta_1 + \delta_2}{2} \cos \frac{\delta_1 - \delta_2}{2}$. It is useful to lay out according to the secondary circle δ_1 or of the uniform globe δ_2 as discussed in the case of projection No. 40; quite similar considerations apply here. If one wanted to imagine the auxiliary-cone as being at such an inclination that it were contiguous with the uniform globe at circle δ_0, where $\cos \delta_0 = n$, then no secondary-circle of the globe would coincide with its representation on the cone.

It seems necessary to point out here some discrepancies in the chapter "Area-true conical projections" ("Flachentreue Kegelprojektionen") in Z.-B.[35] Apparently, there are four different sorts of area-true conical projections treated there:

1. Area-true conical projection on contiguous cones: This ought to be our No. 51 with the equation $\alpha = n\lambda$; $n = \cos \delta_m$; $r_0 = \sec \delta_m - 1$; $r = \sqrt{1 + \sec^2 \delta_m - 2 \sec \delta_m \cos \delta}$. Bludau, however, makes a mistake by unjustifiedly also stipulating $r_0 = 0$. In this manner, he arrives at the false conclusion that "with the selection of the contiguous-cone, the contiguous secondary circle can not be represented true to scale" (p.123). He takes the contiguous-cone of the secondary circle $\delta = 50°$ as an example. Obviously this cone has an $n = \cos 50° = 0.6428$, so that the thusly laid off cone fills a sector of the circle of $360° \cdot \cos 50° = 231.4°$. Instead of this, Bludau calculates $n = 0.50302$, so that by his calculation the sector of the circle would only comprise 181.1°. It is clear that this auxiliary-cone has nothing to do with the secondary circle $\delta = 50°$. Naturally one could draw Bludau's projection with $n = 0.50302$ on the cone he describes, which is contiguous with the uniform globe at circle $\delta = 50°$. But the map would only fill a fraction of a sector $\left(\frac{503}{643}\right)$ of the surface

of the ‹ ‹nd the planes of the prime-points of the globe would not pass through the cone-laterals which are supposed to illustrate them. Actually, Bludau's Projection 1 cannot be considered a contiguous-cone, as it would be according to the title he gives; rather, it should be considered an intersecting-cone; that is, it is not the same as our ring-map No. 51 on a contig‹ous-cone, but, rather, our prime-point map No. 44 on an intersecting-cone.

As his No. 2, Bludau treats "Lambert's area-true conical projection with length-true central parallel." Here, too, the point-map stipulation $r_0 = 0$ is retained, but now the contiguous-cone, which was stipulated under No. 1 but never achieved, is rejected. Thus, No. 2 turns out to be identical to the projection on an intersecting-cone as derived under No. 1, in other words No. 44 of our table. Actually the equations are $n = \cos^2 \frac{\delta_m}{2}$; $r = 2 \sin \frac{\delta}{2} : \sqrt{n}$.

Bludau's projection No. 3 is called *area-true conical projection with least possible shape distortion.* The equations read:

$$n = \cos \frac{\delta_1}{2} \cos \frac{\delta_2}{2} = \cos^2 \frac{\delta_m}{2};\ r - 2 \sin \frac{\delta}{2} : \sqrt{n}.$$

Thus, this is also the same as our No. 44; the exception is that two N, δ_1 and δ_2 are also named along with the scale-true N δ_m; equal-shape and length distortion dominates and the border circles are perceived on them.

Then, No. 4 appears as the projection of Albers, the same as our No. 52.

Actually, then, the projections derived under three different designations, No. 1-3 of Bludau, are all identical with our projection No. 44. The projection No. 41 of our table is attempted, in his No. 1, to be sure, but it is never achieved.

Branch A: Cellular; order b: Curved-Cellular (SNo. 52-58)

Thus far we have exhausted the great order of the straight-cellular true-circular projections. It is understandable that cellular projections with curved H offer little of interest. Only a single projection of such nature has been proposed up to now; it has, however, aside from the great disadvantage of having unilaterally curved H, a number of valuable properties which cannot be found all together in any other type. This is the projection of Wiechel (SNo. 53). It originates on the picture-plane which is contiguous with the globe at the prime-point, when each prime-circle is extended onto the flat surface of the picture without change of its shape and its arrangement around its tangent at the prime point. This projection, then, reproduces all H and all N as equally distributed circles. All H are true to scale. True-shape dominates at the prime-point of the map, and the map is an area-true, centrally-circular and zenithal point-map. Despite the fact that the H are true to scale, the projection cannot be called interval-true, because we take interval to mean the measured straight-line distance between two points on the map's surface, and on this projection the H are semi-circles.

No. 54-58 follow, a few curved-cellular radial-circular projections which have no practical value at all and are only included with regard to the table of terms (p.177 FNo. 38, 41-44).

Branch B: Non-Cellular; Order: Conical (SNo. 59-73)

The true-circular projections with secondary-circles which are not cellular can only be conical, not radial-circular, so that this is the only order which appears in the branch. The property of being conical-radial $\alpha = n\lambda$ (n = constant) has been lost; only the central H $\lambda = 0$ is always assumed to be a

Branch A: Cellular; order b: Curved-Cellular (SNo. 52-58)

Thus far we have exhausted the great order of the straight-cellular true-circular projections. It is understandable that cellular projections with curved H offer little of interest. Only a single projection of such nature has been proposed up to now; it has, however, aside from the great disadvantage of having unilaterally curved H, a number of valuable properties which cannot be found all together in any other type. This is the projection of Wiechel (SNo. 53). It originates on the picture-plane which is contiguous with the globe at the prime-point, when each prime-circle is extended onto the flat surface of the picture without change of its shape and its arrangement around its tangent at the prime point. This projection, then, reproduces all H and all N as equally distributed circles. All H are true to scale. True-shape dominates at the prime-point of the map, and the map is an area-true, centrally-circular and zenithal point-map. Despite the fact that the H are true to scale, the projection cannot be called interval-true, because we take interval to mean the measured straight-line distance between two points on the map's surface, and on this projection the H are semi-circles.

No. 54-58 follow, a few curved-cellular radial-circular projections which have no practical value at all and are only included with regard to the table of terms (p.177 FNo. 38, 41-44).

Branch B: Non-Cellular; Order: Conical (SNo. 59-73)

The true-circular projections with secondary-circles which are not cellular can only be conical, not radial-circular, so that this is the only order which appears in the branch. The property of being conical-radial $\alpha = n\lambda$ (n = constant) has been lost; only the central H $\lambda = 0$ is always assumed to be a

straight line ($g[\delta] = 0$). The equations for these projections are thus $\alpha = \lambda h(\delta)$ and $r = f(\delta)$ where $h(\delta)$ may not be a constant. One calls these projections "improperly-conical" ("unecht konish"/"-kegelig", i.e., conventional-conical). But one cannot in most cases indicate a cone for them which is contiguous with or intersects the uniform globe on which it is appropriate to consider the map projected, since the projection reproduces either all N or none of the N true to scale, and since, on the other hand, the N exhibit almost all different lengths from the lengths of the circles of the cone on which they lie. An exact definition of *improperly-conical* has usually been avoided in the text books until now. With proper conical projections one stipulates:

the property of being conical-radial, that is, $\alpha = n\lambda$ (n = constant) and of being conical, that is, $r = f(\delta)$.

With the projections known to be designated as improperly-conical, the stipulation $r = f(\delta)$ is maintained; however, from the stipulation $\alpha = n\lambda$ only the therein contained stipulation of secondary-circle-distribution is maintained, where n does not have to be a constant but is dependent on δ, not on λ. One might come across the idea of including yet another stipulation that at least one N be true to scale, so that one could consider the map as being on a cone which is contiguous with or intersects the uniform globe at the secondary-circle which has been reproduced completely unchanged. This additional stipulation would be, however, inappropriate. For, if one, for example in the case of the Mercator-Bonne Projection (SNo. 59) which is circle-interval-true and width-true (all the parallels of latitude are true to scale), maintains the intervals of the N but changes their angle-openings at a constant ratio, then if one is to consider the map in its new form as circle-interval-true, no N of the map would be true to scale vis-a-vis the uniform globe on which equidistance exists. Thus, the projection

should not be designated improper-conical. Naturally, one could just as well consider the new map as width-true (all N are true to scale) - when the inclination is earth-axical (i.e. equatorial) - if one coordinates it with a uniform globe with a correspondingly altered radius; then the map would still have to be designated as improper-conical. These discrepancies are put aside only if one rejects the additional stipulation and defines appropriately as follows: *A projection is called "improper-conical" is its N are concentric, equally distributed arcs, and its H are not the radials of these arcs.*

Correspondingly, a projection is called *improper-cylindrical* if its N are parallel, equally distributed straight lines, and its H are not, however, equidistant straight lines perpendicular to the N. A projection is designated as *improper-radial* if its N are concentric equally distributed full circles, but the H are not radials of these circles. An example of an improper-radial projection is the one by Wiechel (SNo. 53); it is cellular, however, whereas the improper-conical and improper-cylindrical projections proposed thus far are not cellular, although they are characterized by a straight-line central H. A cellular improper-cylindrical projection is inserted under SNo. 224.

Class I: Projections with equally distributed prime-lines (SNo. 59-69)

Among the conical-circular - but not the conical-radial - projections the class with straight-line and equally distributed central H appears first. The central H $\lambda = 0$ has, then, the equation $\alpha = 0$ and the distribution $r = m\ (\mathrm{tg}\ \delta_m + \delta - \delta_m)$. The most uniform of its subclasses is the one with *scale-similar* N, whose length remains in a constant ratio to the length of their originals on the uniform globe. Thus, $r\ \alpha = n\lambda\ \sin$, where n is a constant. All these projections are also *area-similar* since their units of area remain at a constant ratio to the originals on the uniform globe.

The subclass is broken up into the groups: ring-maps, which are *point-ring-maps* because of the scale-similarity of *all the N*, ($\delta_m \geq 0$), and *prime-point-maps* ($\delta_m = 0$); and in both groups we obtain equidistance and/or only distance-similarity, according to whether $m = 1$ or $m \gtrless 1$, and we have area-true or area-similar projections, according to whether $mn = 1$ or $mn \gtrless 1$. If $n = 1$, the N are true to scale and the projection is distance-true at an equatorial inclination (bei erdachsiger Lage). The point-ring-map with $m = n = 1$ is the so-called Bonne projection (No. 59), which one should actually call the Mercator-Bonne projection in order to recognize not only its most well-known employment by Bonne (1752), but also its first employment by Mercator (1584). The projection is area-true, circle-interval-true, shape-true along the N δ_m and distance-true at its only used inclination. One can imagine it as lying on the cone which is contiguous with the globe at δ_m; only that circle of the cone which represents the secondary-circle δ_m belongs completely to the unique world map.

If $m = 1$ but $n < 1$ (SNo. 60) the property of being circle-interval-true is maintained, but there is no true-shape or true-scale at any N. Also, the property of being area-true must become instead area-similarity, so that all areas on the map, according to size, are in constant ratio to their originals on the uniform globe. If one wants to consider such a map representation as area-true by means of another choice of uniform-globe (SNo. 61), then its property of being circle-interval-true must become circle-interval-similarity, since then $m = 1 : n$ different than 1 must be assumed. In another instance, such a projection where $n = 1$ and $m \gtrless 1$ (SNo. 62) would show all N as true to scale, but is would nowhere be area-true, shape-true or circle-interval-true.

In the group of *prime-point-maps* ($\delta_m = 0$) the case $m = n = 1$ presents the projection of Stab (also called the Stab-Werner Projection, since Werner completed it at the suggestion of

Stab in 1914) (SNo. 63); it is a special case of the Mercator-Bonne projection No. 59 with all the properties stated there, where true-shape exists only at the prime-point $\delta_m = 0$.

We find the case $m = 1$; $n < 1$ (in the exception $n = 1/2$) the same as Schjernings earth map No. 5 at its equatorial inclination in Z.-B.[36] One cannot designate the projection as being area-true and at the same time central-interval-true, as happens in the above source. If one lets the intervals of the parallels of latitude on such a map agree with those on the uniform globe ($m = 1$), then the areas and the arcs of the parallels of latitude are only half as great as on the globe; then this projection No. 64 is central-true, but only area- and distance-similar. If one is to consider the map area-true, however, and assumes thus that $mn = 1$ (No. 65), then there is only similarity among the intervals and distances, not equality, that is, $m = \sqrt{2}$ and $n = 1 : \sqrt{2}$. If the Schjerning map is to be considered as distance-true, that is, $n = 1$ (No. 66), then the intervals and areas are twice as great as on the globe; $m - 2$ and $mn - 2$. Both projections, No. 65 and 66, are thus not to be included in the group of central-interval-true projections.

Interval-similar projections with any rule other than scale-similar N have hardly been proposed. Only central-interval-true maps are yet to be found, on which the entire length of the arc of the N is stipulated by means of the special form of the borders of the world map. On Schjernings oblique-axical world map No. 3 (SNo. 67) this world border consists of two full circles which are contiguous at the prime-point (London). The equations which result for them are $r = \delta$; $\alpha = \lambda/\pi \text{ arc cos } \frac{\delta}{\pi}$. The N $\delta_m = 79°27'$ is true to scale, where $\frac{\delta_m}{\pi} \doteq \cos\left[\frac{\pi \sin \delta_m}{2\delta}\right]$. True-shape exists at the five points $\lambda = 0°$ or $180°$, $\delta = 0°$, $180°$ or $79°27'$.

On Schjernings transverse-axical map No. 2 an ellipse is to be considered the border-curve of the left half of the map (SNo. 68); the prime axes of the ellipse are equator and central meridian in scale-true reproduction. Thus, the equations

$$r = \delta \text{ where } \delta < \frac{\pi}{2}; \alpha = \lambda, \text{ where } \delta > \frac{\pi}{2}; \alpha = 2\beta\lambda : \pi,$$

where $tg^2\beta = (5\pi^2 - 4\delta^2):(4\delta^2 - \pi^2)$. The right half of this world map (SNo. 69) has a delimitation which was not stated analytically. It does not deviate very much from the following equation

$$r = \delta \text{ where } \delta < \frac{\pi}{2}; \alpha = \lambda, \text{ where } \delta > \frac{\pi}{2}; \alpha = 2\beta\lambda : \pi,$$

where $\beta = \pi - \delta$. The map which Schjerning has thus constructed belongs in our Family V, since the two halves of the map are drawn according to different rules. It is hard to understand why Schjerning did not grant the improvement made on the right half of the map, North Asia, also to the left half of the map, Northwest America.

Of all the maps in the class (No. 59-69) one may of course only consider No. 59 as drawn on the contiguous-cone at the secondary-circle δ_m which is distinguished by its true-shape. In the case of all prime-point-maps, the auxiliary-cone obviously passes over to the contiguous-plane of the basic-point, and this plane is also the one accepted for the ring-maps No. 60-62, since no distinguished auxiliary-cone can be given.

CLASS II: With Unequally Distributed Prime Lines (SNo. 70-73)

The *central-interval-true* projections of the preceding subclass were determined by the form of the boundary lines of the world map, the H as $\lambda = \pm\pi$; this can also be done for the area-true

conical projections, divided according to secondary circles, in which case the interval-similarity is lost. Thus we obtain the single subclass and group of class II.

Nell[37] was the first to propose such a projection (1890). He chose as border line the mean between that of the area-true, true-conical projection with shape- and scale-true contiguous-circle $_m$ (SNo. 51) and the likewise area-true, improper-conical projection of Mercator-Bonne (SNo. 59), which reproduces the same contiguous-circle δ_m shape- and scale-true. Nell carried out this projection, mediating between the two, as an ellipsoid. Hammer[38] reported in 1900 on Nell's projection and attempted to derive an exception for $\delta_m = 90°$ and for the globe. But Hammer had misunderstood Nell's way of thinking; and thus this projection by Hammer (SNo. 137) is not a special case of Nell's projection, which is for this reason more thoroughly explained below.

If μ is the entire curvature of the arc of an N of the map, so that $\alpha:\mu = \lambda:2\pi$, then in projection 51, $\mu(51) = 2\pi \cos \delta_m$ and in projection 59, $\mu(59) = 2\pi \sin \delta:r$, where $r = \text{tg}\, \delta_m + \delta - \delta_m$. Now, Nell determines the delimitation of his map by finding the mean of the two values $\mu(51)$ and $\mu(59)$ *with r having the same value*. The misunderstanding of Hammer was that he believed that Nell wanted to find the mean of the two μ-values *using the same value for* δ. For a correct understanding of Nell's ideas, it is first necessary to express $\mu(59)$ with r rather than with δ, i.e. $\mu(59) = 2\pi \sin (r - \text{tg}\, \delta_m + \delta_m):r$; and then the mean of $\mu(59)$ and $\mu(51)$ may be found. Thus one obtains for the Nell Projection (SNo. 70): $\mu = \frac{\pi}{r}\,[r \cos \delta_m + \sin (r - \text{tg}\, \delta_m + \delta_m)]$. This stipulation, then, results in true-area for each globe-dome between $\delta = 0$ and δ with the corresponding ring-section between r_0 and r on the map:

$$\int_{r_0}^{r} \mu r\, dr = 2\pi\,(1 - \cos \delta) = 4\pi \sin^2 \frac{\delta}{2}\,,$$

which, when integrated results in the basic equation

$$\frac{r^2}{2} \cos \delta_m - \cos(r - \text{tg } \delta_m + \delta_m) - \frac{r_0^2}{2} \cos \delta_m + \cos(r_0 - \text{tg } \delta_m + \delta_m) =$$

$$2\pi(1-\cos\delta).$$

Then, $r_m = \text{tg } \delta_m$ for No. 51, as well as for No. 59, and hence, also for the Nell Projection. True-scale and true-shape and the maximum value for μ exist at this N. Thus, one must stipulate r_0 in such a manner that the basic equation is accomplished when one employs $\delta = \delta_m$ and $r = r_m = \text{tg } \delta_m$. This results in:

$$r_0^2 \cos^2\delta_m - \cos(r_0 - \text{tg } \delta_m + \delta_m) = 1 - 4\cos \delta_m + \cos^2\delta_m.$$

I have carried out the calculation for the example $\delta_m = 45°$ under SNo. 71 $r_m = 1$; tg $\delta_m = 1 = 57°17,75'$. The basic-equation becomes:

$$r_0^2 : \sqrt{8} + \cos(r_0° - 12°17.75') - 0.93935.$$

One may find by trial that $r_0 = 21°15' - 0.37088$, so that the equations for the Nell Projection read:

$$\cos \delta = 0.53032 - r^2 : \sqrt{32} + 0.5 \cos(r° - 12°17.75');$$

$$\mu° = 180° : \sqrt{2} + \frac{180°}{r°} \sin(r° - 12°17.75'); \ \alpha = \mu\lambda : (2\pi).$$

(The forms $r°$ and $r_0°$ mean that r and r_0 are to be expressed in degrees as arcs of the uniform globe.) One can calculate a table of δ and μ to r-values, draw two curves which represent r and μ as functions of δ and then - if necessary, running a mathematical check according to the projection-equations - obtain the r- and μ-values to rounded δ-values. The following table gives the values of r and μ to rounded δ-values for an area-true improper-conical Nell Map with scale-true secondary-circle $\delta_m = 45°$. r is stated as a fraction of the radius of the uniform globe, as well as in arc degrees ($r°$).

Radius r and Angle Openings μ of the Secondary Circles of an Area-True Nell Projection with Scale- and Shape-True Secondary Circle $\delta_m = 45°$ (SNo. 71)

[The above noted table may be found on page 51a following.]

Projection No. 72 gives the special case of the Nell Projection when $\delta_m = 0$, where the cone becomes plane and the map becomes a prime-point-map. This map is an intermediary between the radial, area-true projection of Lambert (No. 23) and the one by Stab (No. 63). Here $\mu° = 180° \left[1 + \frac{\sin r}{r}\right]$ and since it is obvious that $r_0 = 0$, the relations between r and δ must read $\cos\delta = \cos^2\frac{r}{2} - \frac{r^2}{4}$

The following table gives correlated values of δ, r and μ.

Radius r and Angle Openings μ of the Secondary Circles of an Area-True Prime-Point Nell Map (SNo. 72). $r_0 = 0$

[The above noted table may be found on page 51a following.]

On this map all the N are greater than on the globe and true-shape exists only at the prime-point $\delta = 0$.

SNo. 73 is another example of such a conic, area-true, prime-point map, on which the border of the world map is arbitrarily prescribed; it consists of a semicircle with radius R at the prime-point of the map (c.f. Illustration 3, plate 1) and two semicircles with radius $\frac{R}{2}$ which connect with no break. The property of entire true-area is stipulated with $R = 4:\sqrt{3}$. If the secondary circle δ is to be reproduced by means of an arc with radius r and the angle-opening μ = 2 times $\measuredangle$ BMH, then the triangle MBA results in $r - R \sin \pi/2$. The globe-dome delimited by secondary circle δ has the area $2\pi(1 - \cos\delta)$, and on the map it consists of a circle-segment with radius r and the angle-opening μ (Content $r^2\mu/2$) plus 2 times the circle-segment above the chord MB = r. The radius of the segment-circle is $OM = R/2 + 2/\sqrt{3}$

Halbmesser r und Winkelöffnungen μ der Nebenkreise eines Nellschen flächentreuen Entwurfs mit maß- und winkeltreuem Nebenkreis $\delta_m = 45°$ (SNr. 71)

δ°	r	ε°	μ°	δ°	r	ε°	μ°	δ°	r	ε°	μ°
0	0,3709	31° 15,0′	202,3	40	0,9127	53° 17,5′	254,0	110	2,0134	117° 31,0′	212,1
5	0,3880	23 14,0	207,3	45	1,0000	57 17,5	254,6	120	2,1754	124 18,7	203,6
10	0,4317	24 54,8	217,7	50	1,0873	62 17,8	254,1	130	2,2935	131 24,5	195,8
15	0,4984	28 36,8	225,6	60	1,2605	72 13,2	250,9	140	2,3916	137 1,9	190,1
20	0,5758	33 57,8	237,7	70	1,4308	81 58,8	245,3	150	2,4715	141 36,3	183,6
25	0,6560	37 35,1	244,5	80	1,5969	91 29,8	238,0	160	2,5293	144 56,5	179,0
30	0,7401	43 24,4	249,3	90	1,7550	100 35,3	229,8	170	2,5651	146 50,8	177,2
35	0,8261	47 10,9	252,4	100	1,9074	109 17,3	228,0	180	2,5775	147 40,6	176,3

Halbmesser r und Winkelöffnungen μ der Nebenkreise einer Nellschen flächentreuen Hauptpunktkarte (SNr. 72). $r_0 = 0$

δ°	r	ε	μ°	δ°	r°	ε°	μ°	δ°	r	ε°	μ°
10	0,1745	10° 00,0′	359,1	70	1,1826	67° 30,0′	321,2	130	1,9518	112° 00,0′	265,1
20	0,3490	20 00,0	356,4	80	1,3329	76 22,3	311,3	140	2,0393	116 52,9	258,7
30	0,5201	29 49,2	352,7	90	1,4753	84 43,0	301,3	150	2,1075	120 45,9	252,4
40	0,6908	39 31,8	346,0	100	1,6134	93 30,0	291,4	160	2,1570	123 31,8	248,5
50	0,8592	49 10,3	338,7	110	1,7403	99 43,8	281,9	170	2,1871	125 19,6	247,2
60	1,0215	58 31,8	330,3	120	1,8546	106 15,6	273,2	180	2,1971	125 52,8	266,6

of the segment-angle MOB = $2\pi - \mu$, thus, the area of the segment = $\frac{2}{3}[2\pi - \mu + \sin\mu]$. The equation between μ and δ is then: $2\pi(1 - \cos\delta) = r^2 \frac{\mu}{2} + \frac{4}{3}[2\pi - \mu + \sin\mu]$; by employing $r^2 = \frac{16}{3}\sin^2\frac{\mu}{2}$ and simple conversion, this may be stated $\cos\delta = \frac{2}{3\pi}(\mu\cos\mu - \sin\mu) - \frac{1}{3}$

μ is always an over-obtuse angle; when $\delta = 0$ then $\mu = 2$, when $\delta = \pi$ then $\mu = \pi$. The equations for the projection are, then:

$$\cos\delta = \frac{2}{3\pi}(\mu\cos\mu - \sin\mu) - \frac{1}{3};\quad r = \frac{4}{\sqrt{3}}\sin\frac{\mu}{2};$$

$$\alpha = \frac{\lambda\mu}{2\pi} = \lambda\mu^\circ:360^\circ.$$

The following table gives correlated values for δ, r and μ.

Radius r and Angle-Opening μ of the Secondary-Circle of an Area-True Prime-Point Maurer Map (SNo. 73) $r_0 = 0$; $\mu_8 = 360°$

[The above noted table may be found on page 52a following.]

The secondary-circle δ_m is true to scale $\mu° = 360°\sin\delta$ for the r. This is the case for $\delta_m = 45°46'$ where r = 0.8088 and $\mu° = 319°$. True-shape exists at the points $\lambda = 0°$ or 180° and $\delta = 0°$ or 45°56'.

The projections No. 71-73 should not be considered trifles, since they display certain advantages over the projections of Lambert (No. 23), Mercator-Bonne (No. 59) or Stab (No. 63). They are to be preferred over the older projections for the shape in their margins and, in general, for the arc-lengths of the H. In No. 23 all H are reduced at a ratio of $2:\pi$ 0.637; in No. 63 all are increased, the ratio of increase growing from 1.0 at the central-H to about 1.9 at $\lambda = \pm\pi$. In No. 72 (Nell), all H are reduced also, but less than in the Lambert projection (No. 23); the ratio of reduction varies between 0.695 and 0.877; and in

Halbmesser r und Winkelöffnungen μ der Nebenkreise einer Maurerschen flächentreuen Hauptpunktkarte (SNr. 78). $r_0 = 0$; $\mu_0^\circ = 360^\circ$

δ°	r	τ°	μ°	δ°	r	τ°	μ°	δ°	r	τ°	μ°
10	0,1760	10° 4,02'	351,23	70	1,2197	69°52,53'	290,21	130	2,0333	116° 30,0[illegible]'	236,27
20	0,3331	19 6,03	342,40	80	1,3793	75 20,10	253,63	140	2,1297	121 55,02	225,[illegible]3
30	0,5311	30 25,71	333,41	90	1,5222	87 47,40	270,37	150	2,2012	126 17,[illegible]	214,73
40	0,6985	39 26,52	324,34	100	1,6750	96 0,21	226,97	160	2,2510	129 32,46	203,57
50	0,9223	50 33,06	315,03	110	1,5085	103 37,14	250,91	170	2,2067	131 33,21	192,[illegible]
60	1,0332	60 22,60	305,70	120	1,9250	110 21,32	246,71	180	2,3031	132 10,0[illegible]	[illegible]

No. 73 (Maurer) some H are less, some greater than on the globe, the ratio of distortion changing from 0.735 at the central-H to 1.155 at the margin-line $\lambda = \pm\pi$.

Family I, Branch B: True-Circular, Not Divided according to Secondary-Circles (SNo. 74-82)

The true-circular projections not divided according to secondary circles, which can be radial (subbranch A) or conical (subbranch B), are of secondary significance. A few of them have been mistakenly designated as zenithal, in that zenithal has been equated with central-circular without stipulating the essential property of being divided according to secondary-circles. For the same reason such projections should also not be called improper-radial or improper-conical (c.f., p.44). Just to make these differences clearly understandable, the otherwise quite worthless projection No. 74 was presented in 1914 in subbranch A of the *radial-circular projections*.[39] It is central-interval-true with concentric full circles as N and also has the rays of these full circles as H. But it deviates from being zenithal and radial in that these H do not intersect themselves at that angle $\alpha = \lambda$ but at the angles $\alpha = \pi \sin \frac{\lambda}{2}$. True-shape, then, does not exist at the prime-point of the map; but in agreement with the condition $d\alpha : d\lambda = \sin\delta$ along a curve, for which the equation in polar coordinates reads: $2 \sin r = \sqrt{\pi^2 - \alpha^2}$.

The two following projections, No. 75 and 76, with the same H, have also been included only for the sake of the table of terms.

Projection No. 77 is center-interval-true, not only in relation to one point, with the two points A and B; it is, then, doubly-interval-true. The source of the coordinates on this map would be the central-M of the line-segment 2L between the points A

and B which incline to the y-axis. We measure the coordinate λ of the globe on the corresponding great-circle from the original of point M and, perpendicular to that, the coordinate ϕ. Then the equations for the doubly-interval-true projections are:

$$x^2 + (L + y)^2 = \delta_1{}^2 \quad \text{and} \quad x^2 + (L - y)^2 = \delta_2{}^2$$

$$\text{where } \cos \delta_1 = \cos \phi \cos (L + \lambda); \cos \delta_2 = \cos\phi \cos (L - \lambda).$$

Thus one obtains: $y - (\delta_1{}^2 - \delta_2{}^2):(4L)$ and

$$x = \sqrt{\frac{\delta_1{}^2 + \delta_2{}^2}{2} - L^2 - y^2}$$

In 1919[40] I explained in more detail how one can use this type of projection, for example, to be able to read on a map, true to scale, by how many sea-miles the distances from one place in the Atlantic to a place in the Pacific differ when traveling via Panama, Cape Horn or the Magellan Strait.

The following projections, No. 78-80, are counter-azimuthal, that is, the azimuth-equivalents extending toward a certain point of the map are straight lines which intersect themselves at this point at proper angles. An azimuth-equivalent on the globe connects all points from which a certain aiming point lies on the same azimuth A. More detail about this is given for the azimuth-equivalent map, SNo. 189, in which not only the straight lines through a single point, but all the straight lines of the map surface, are pictures of azimuth-equivalents. The stipulation of only one counter-azimuthal aiming-point S is not sufficient for a projection. Therefore, a further stipulation can be made. E. Hammer[41] added the stipulation of axiality and presented a central-interval-true, counter-azimuthal projection (No. 78). If the aiming-point S of the azimuth-equivalent has, on the globe, the geographic latitude $\phi = \phi_0$ and the longitude $\lambda = 0$, and if one calculates the azimuth A from the north above east to 360°, then the equation for the

azimuth-equivalents on the globe is $\cot A = \operatorname{tg} \phi_0 \cos \phi \operatorname{cosec} \lambda - \sin \phi \cot \lambda$.

If the aiming point S on the map is $x = y = 0$, and if its meridian is the straight line $y = 0$, then the picture of the azimuth-equivalent on a counter-azimuthal projection must be the straight line $x = y \cot A$. If the projection should also be central-interval-true, then $x^2 + y^2$ must equal r^2 and $r = \delta$, where δ is the arc-interval of the point (ϕ,λ) of the aiming point $(\phi = \phi_0;\ \lambda = 0)$, thus

$$\cos \delta = \sin\phi \sin \phi_0 + \cos \phi \cos \phi_0 \cos \lambda$$

The Hammer Projection is determined with these equations. For it $x = -\delta \cos A$; $y = -\delta \sin A$, if x is calculated positive toward north and y positive toward east. The scale-true intervals are calculated here from the counter-azimuthal point S. The cylindrical-circular projection SNo. 215 in Family IV shows that one can combine counter-azimuthality - a property of the map, but not of its basic grid - with true-distance (a property of ths grid!) where a point other than the counter-azimuthal point is concerned. (More detail about this also on P. 85)

One can make r = another function of δ, rather than $r = \delta$, on a true-circular, counter-azimutahl projection, but it does not make much sense to do so. Hammer's projection very appropriately shows every map-point and in which direction and which distance from that map point the aiming point lies (e.g. Mecca). Instead of $r = \delta$, Bludau has employed $r = \frac{1}{2} \operatorname{tg} \frac{\delta}{2}$ (SNo. 79), as in the case of the true-shape projection (SNo. 2), but without accomplishing true-shape. True-shape, central-circular maps cannot be counter-azimuthal. Bludau[42] calls his and Hammer's projection zenithal, but mistakenly, since they are not divided according to secondary circles. One could also employ $r = 2 \sin \frac{\delta}{2}$ (SNo. 80) and in this manner still maintain equal-area,.

(but not true-area!) circle-shaped zones around aiming-point S of the counter-azimuthal map. Maurer[43] has shown that a true-area, counter-azimuthal projection is possible. But it cannot be central-circular and belongs to Family IV (SNo. 219) because of its reduced symmetry ratios.

In subbranch B of the conical projections the full circularity of the N is dropped. Such a projection has been used to a great extent, the so-called *simplified conical projection* (SNo. 81), which has been incorrectly attributed to Mercator but was first presented in 1745 by De l'Isle. It is circle-interval-true and the radii of the N are, the same as on the ordinary conical projection No. 47, on a contiguous-cone at secondary circle δ_n, thus $r - tg\ \delta_n - \delta_n + \delta$. Now, this N is not reproduced true to scale, but, rather, the N-pair $\delta_1 = \delta_n + \varepsilon$ and $\delta_2 = \delta - \varepsilon$. Thus, for N δ_1 $r_1 = tg\ \delta_n - \delta_n + \delta_1$; $\alpha_1 = \frac{\lambda \sin\delta_1}{r_1}$ and for N δ_2 $r_2 = tg\ \delta_n - \delta_n + \delta_2$ and $\alpha_2 = \frac{\lambda \sin\ \delta_2}{r_2}$. The H are straight lines through the dividing points on both N δ_1 and δ_2, which divide no other N regularly and also do not pass through a point. Only the two points $\lambda = 0$, $\delta = \delta_1$ and $\delta = \delta_2$, are true-shape.

One could also determine the radii of the N as in SNo. 81, dividing the N δ_n, except for δ_1 and δ_2, true to scale, and then lay the H as circle-arcs through the three at δ_1, δ_2 and δ_n, but this would be less convenient to draw. Also, such a map would in general not have to be expanded to the point of being a world map. Nevertheless, this projection plays an important role in the history of map projection. Actually, it is the *second projection of Ptolemaus* (130 A.D.). Ptolemaus used the three parallels of latitude $\delta_1 = 27°$ (Thule), $\delta n = 66°10!$ (Syene) and $\delta_2 = 106°.5$ (Meroe), as is shown in Herz' description.[45] D'Avezac, German and Gretschel assumed inappropriately that Ptolemaus had made not only the three, but all the parallels of latitude true to scale, in which case the second projection of Ptolemaus would be identical with the Mercator-Bonne projection (No. 59). The fact that no world map can be drawn according to

the projection of Ptolemaus is because, with a small difference in length λ from the straight-line central meridian, the H-circles do not at all intersect the N-circles outside the zone between δ_1 and δ_2, and no reproduction of the concerned sections of the globe can be obtained. Also, the two infinitely adjacent H, which intersect themselves, do not do so at all on the N-lines which are pictures of the two poles of the earth, but rather at other parallels of latitude; this is just incompatible with an illustration. On the central meridian one can only go so far until the H-lines touch the N-lines, or until, with increasing λ, the length of the arc on N begins to decrease. *Illustration 4* (Plate II) gives the right half of the Ptolemaus Grid in interstices of 15° in width and length (the numbers 1-12 for λ mean multiples of 15°); it shows - e.g., at $\delta = 15°$ ($\phi = 75°N$) - that one cannot even go over $\lambda = 45°$ on account of the reversal of the H-segments, and that, even on the scale-true N-circle $\delta_1 = 27°$ ($\phi = 63°N$), which is the basis of the projection, one cannot go out past $\lambda = 150°$, because the H touch the N-circle and/or extend to the wrong side of it. Of the middle points - indicated with χ - of the H-circles (5) to (12), the middle point (10) lies on the straight line of the center S of the N through the dividing-point 10 at δ_1, thus the H-circle 10 already touches the N-circle δ_1, while the H-circles 11 and 12, which extend from their aiming-points to δ_1, give way toward lesser values, rather than going toward the greater. Even the H-line 1 ($\lambda = 15°$) reaches neither the circle $\phi = 90°$ nor the circle $\phi = -75°$. Thus, this projection can only reproduce very small parts of the earth outside of zone δ_1 and δ_2, and not even this zone in entirety.

Family II, Branch A: Straight-Symmetrical, Divided According to Secondary Circles (SNo. 83-152)

While the projections of Family I were *centrally-circular* but lacked in *double-symmetry*, there follow in Family II the

straight-symmetrical projections, which combine both properties, in that the N become parallel straight lines [$x = F(\phi)$] and the map picture becomes symmetric to the base-line $x = \phi = 0$, as well as to the central-H $y = \lambda = 0$. All these projections are *cylindrical-circular*. Here also, we differentiate between a branch A, divided according to secondary circles, and a branch B, not divided according to secondary circles. The stipulation of symmetry to the central-H makes projections with curved cells impossible here. All cellular projections (subbranch A) can only be straight-cellular (order a). Ordered in this manner, all H are equal-interval, straight lines perpendicular to the N, and can be considered as lateral lines of a straight cylinder. These projections are thus not only improper-cylindrical [$x = F(\phi)$] but are also cylindrical-radial [$y = n\lambda$], whereby they become *cylindrical* or proper-cylinder projections (class 1). Thus, improper-cylindrical + cylindrical-radial = cylindrical, if, as here, the infinitely distant intersecting-point of the cylinder-rays is at the same time the middle point of the parallel N-straight-lines. That is, in a cylindrical projection, the H-parallels and the N-parallels intersect perpendicularly. This must be emphasized, because reproductions are possible, which are cylindrical-radial or improper-cylindrical, but which are not cylindrical. An example of this is SNo. 224, a perspective picture of the Mercator map. Its H and N are many parallel straight lines, which, however, do not intersect at right angles. Projection 224 is even straight-cellular; for its H are equal-interval, parallel straight lines and its N equal in distortion. It is cylindrical-radial and can be laid onto a straight cylinder in such a manner that its N fall on the cylinder-circumferences. But it is not cylindrical; for, the map cannot be projected onto a straight cylinder in such a manner that the H fall on the lateral lines and the N onto the circumferences of the cylinder at the same time.

Subbranch A: Cellular; Class I: Cylindrical (SNo. 83-97)

When $n = 1$ in the equation $y = n\lambda$, one obtains *contiguous-cylindrical* projections on which the cylinder touches the uniform globe at the scale-true reproduced basic-circle; when $n > 1$, *intersecting-cylindrical* projections with two scale-true N $\pm\phi_m$, where $\cos \phi_m = n$; when $n > 1$, *free-cylindrical* projections, in which all N are greater than scale true. In the subclass of the cylindrical *prime-point maps* all H extend into infinity; on cylindrical *line-ring-maps* they all have finite length.

Among the *cylindrical line-ring maps* (SNo. 83-91) (subclass A, F [±90°] finite), we find the *circle-perspective* projections (SNo. 83-86). In these, each H is reproduced perspectively on the cylinder with a radius R, which is coaxial with the uniform globe, from a point of vision, which has an extreme interval $(1 + q)$ on the intersection-line of the basic-circle- and prime-circle-plane from the prime-circle-line. The general equation for these projections is:

$$x = \sin\phi \, \frac{q + R}{q + \cos\phi}, \qquad x = R\lambda = n\lambda$$

True-shape exists for the secondary-line $\phi = \phi_w$, for which $\cos\phi \, \frac{dx}{d\phi} = \frac{dy}{d\lambda}$. If one employes $(q + R) : R = a$, this results in $\cos \phi_w = \frac{a - 2q \pm \sqrt{(a - 2q)^2 - 4q^2(1 - aq)}}{2\,(1 - aq)}$

Whereas this equation could produce true-shape at 4 N, this cannot be accomplished with the two circle-perspective projections proposed singly above. Both stem from Braun (1867) and are considered as the contiguous-cylinders $R = 1$, thus $a = 1 + q$.

In SNo. 83, *Braun's stereographic cylindrical projection*, $q = 1$, thus $a = 2$, with point of vision on the basic-circle. The equations are: $x = 2 \operatorname{tg} \frac{\phi}{2}$; $y = \lambda$. One finds $\cos \phi_w = \pm 1$, thus, true-shape only on the basic-line which is at the same time true to scale.

SNo. 84 is *Braun's modified Mercator projection,* for which $q = 0.4$ and $a = 1.4$ is considered valid. The equations are $x = \frac{1.4 \sin \phi}{0.4 + \cos \phi}$; $y = \lambda$. True-shape exists for the three N-lines $\phi_w = 0°$; $\phi_w = \pm 68°40.6'$. Only $\phi_m = 0°$ is true to scale.

In No. 85 and 86 I have proposed improvements for No. 83 and 84. An *intersecting-cylinder* ($R = \cos \phi_m$) is used in No. 85. The equations are: $x = (1 + \cos \phi_m)\ \mathrm{tg}\ \frac{\phi}{2}$; $y = \lambda \cos \phi_m$, and now one finds true-shape and true-scale on two N-lines $\pm\phi_m$, rather than only on the basic-line. A free-cylinder ($R = 1.04$) is used in No. 86, which, when $q = 0.4$ is maintained, leads to quite fortunate distribution of the now four shape-true N. The equations for the projection are: $x = 1.44 \sin \lambda : (0.44 \cos \phi)$; $y = 1.04\ \lambda$, and results in $a = 1.3846$; $\cos \phi_w = (0.58462 \pm 0.23700) : 0.89230$, from which one finds $\phi_w \pm 22°57.5'$ and $\phi_w = \pm 67°4.1'$. When compared directly with the uniform globe, all N are of course too great; in choosing the scale, however, one may, maintaining true-shape on the four above-mentioned N, consider yet two more arbitrary N $\pm\phi_m$ as true to scale.

The *contiguous-cylinder-map* of Lambert, SNO. 87, is in the group of the *area-true* ring maps. They can be called zone-perspective, since, according to their law $x = \sin \phi$, every secondary-circle is projected on its plane from the axis onto the cylinder. This area-true cylindrical map is also called an isocylindrical projection. No. 88 gives the generalized form for the various intersecting-cylinders, and No. 89 gives the special case with the least average uppermost shape distortion, proposed by W. Behrmann (1910).

The *circle-interval-true* cylindrical maps are the quadratic (No. 89) and the *rectangular flat map* (No. 90). The latter was invented around 100 a.d. by Marinus von Tyrus; the former, at the earth-axical inclination, has been used since about 1500 (1527 by Thorne), and at transverse-axical inclination by Cassini (1745).

With cylindrical area-true or interval-true (definition on P. 7) projections, true-shape and/or being interval-true is maintained, if one leaves the H unchanged but substitutes the parallel N-straight lines without changing their intervals with parallel curves. The symmetry to the basic-line is, of course, thereby lost, so that the projections then belong to family IV, in which four such projections - SNo. 201-204 - whose N are equally long and parallel arcs, are listed.

In the following subclass of cylindrical prime-point-maps (SNo. 92-97), true area is impossible, since the maps, band width invariable, are infinitely long. As first group we find the *cylinder-perspective* projections, perspectives of the globe on a cylinder projected from point of vision A on the axis of the cylinder. If A lies about the segment q below the basic-circle, then the result of reproducing on an intersecting-cylinder through the secondrry circles $\pm\phi_m$ is generally:

$$x = \cos \phi_m [\sin\phi + q(1 - \cos \phi)]: \cos \phi; \; y = \lambda \cos \phi_m \quad \text{(SNo. 92).}$$

Symmetry to the basic-line $\phi = 0$ appears only in the special case $q = 0$; this is the only example of this thus far proposed. This case from family II is found as the perspective on contiguous-cylinder (No. 93) proposed by Wetch, whereas the general case with $q > 0$ on intersecting- and contiguous-cylinder is found in family IV as SNo. 197 and 198.

Among the cylindrical point maps which are not perspective, the only important one is that determined by the stipulation for true-shape, the Mercator projection; it can be considered contiguous-cylindrical (SNo. 94) as well as intersecting-cylindrical (SNo. 95), according to whether one declares the basic-circle $\phi = 0$ or two other secondary circles $\pm\phi_m$ as true to scale. The world map by G. Mercator appeared in 1569; a sundial of the Nuremberger cartographer E.Etzlaub, made in 1513, shows, however, the same map grid already, so that one should really take up Etzlaub's name in its designation,

perhaps *"Etzlaub-Mercator-Projection."* This is to be recommended, since there are several more different sorts of Mercator projections; Mercator should be considered their originator, but they are mostly not named after him; e.g., Mercator's central-interval-true projection (1569) which is usually named after Postel (1581), Mercator's area-true projection (1603) which is mostly named after Sanson (1650) or even after Flamstead (1700). In our system, the so-called Mercator map is designated sufficiently as shape-true cylindrical projection. The use of his grid in a transverse-axical inclination stems from Lambert (1772).

The equations for the Etzlaub-Mercator-Projection are:

$$y = n\lambda;\ x = n \log \text{nat tg} \left(45° + \frac{\phi}{2}\right);\ n = \cos \phi_m,\ \text{thus, } n = 1$$

with a scale-true basic-circle $\phi_m = 0$. The equation for x results directly from the condition of true-shape of a contiguous-cylindrical projection with straight H $(y = \lambda)$ $\frac{dx}{dy} = \frac{d\phi}{d\lambda \cos \phi}$, thus $\cos \frac{dx}{d\phi} = \frac{dy}{d\lambda} = 1$, from which, by integrating, follows: $x = \int_0^x dx = \int_0^\phi \sec \phi \, d\phi = \log \text{nat} \left(45° + \frac{\phi}{2}\right)$

The author of this paper has given, in 1906,[6] an elementary derivation of the Mercator function x without integral calculation and without approximation to the progression $(\sec \frac{1'}{2} + \sec \frac{3'}{2} + ..)$ but this has remained rather unknown. In that paper the equation for the *equal-course* (loxodrome), that curve, which intersects every meridian at a constant angle α on the globe as well as the earth-axical, true-shape, all-circular projection (SNo. 2), is given; it is derived only under the assumption of the well-known basis of the logarithmic term, that $(1 + \frac{1}{z})^z = e =$ the basis of the natural logarithms, if z extends out past every border. The equation for the equal-course (loxodrome) is found in the form $\lambda \cot \alpha = \log \text{nat tg} (45° + \frac{\phi}{2})$, where α is the constant course-angle of the equal-course, ϕ and λ are

width and length of its continuous point, and λ is calculated from the course-line with the equator. On the true-shape cylindrical projection with parallel H ($y = n\lambda$), the equal-course, which intersects them all at a constant angle α, must be a straight line with an equation

$$\cot \alpha = \frac{x}{y} = \frac{x}{n\lambda} = \frac{\log \text{nat tg} \left(45° + \frac{\phi}{2}\right)}{\lambda},$$ with which the equation for the Etzlaub-Mercator-Projection

$$x = n \log \text{nat tg} \left(45° + \frac{\phi}{2}\right)$$ is derived.

It should be noted here that the cylindrical projections from a cylinder onto a plane can, in this one, represent the surface of the globe infinitely many times repeated. In order to obtain a single reproduction of the world, it is not necessary to choose two straight lines which are perpendicular to the equator as world-borders and would show here a difference in longitude of 360°. One can just as well take two congruent, equivalent curves through two such equatorial points as world-borders. This becomes especially evident in the Etzlaub-Mercator map, when two straight lines which intersect the equator obliquely - i.e., equal-courses of another course-angle - rather than the usual border meridians (equal-courses with course-angle $\alpha = 0$), are drawn. By means of this picture, one frees himself from the nebulous notions that have led to such peculiar, mystical statements about the course-lines; such as: "For the loxodrome, the pole is an asymptotic point, which it is continuously approaching, up to the smallest of small separation, but, despite finite length of all infinitely many turnings, never actually entirely reaching it."[47]
With our world map with oblique equal-course world-border, one sees directly that the course-length of the equal-course extending from course-angle α between the parallels ϕ_1 and ϕ_2 equals $(\phi_2 - \phi_1) \sec \alpha$, and that the Etzlaub-Mercator map increases every course-line-arc in exactly the same ratio as the latitude difference of its endpoints. At the same time,

it becomes clear that every point of the infinitely distant straight line of the map's surface should be included in the picture of the earth-pole as endpoint of the equal-course of a particular course-angle. One can thus show, as was mentioned on p.35, that this true-shape map exhibits no unique, non-true-shape points. To the equal-course with a course-angle of 0° belong not only the meridian arc, which extends into the pole of the globe, but also the infinitely many revolutions around the pole, which, to be sure, have a radius and a course-length of zero but have tangents in all directions; according to appearances, this equal-course converges with only one meridian, but in reality it forms the course-angle 0° with all other meridians. The author discussed this question in more detail in 1919 and 1926.[48]

Projection No. 96 stems from the Etzlaub-Mercator map, if one alters the s- and y-coordinates in different ratios m and n, thus recasting the map as affine. It remains then a *course-line* map, *but is no longer true-shape.* Because of the often heard mistaken statement that the Etzlaub-Mercator is the only reproduction which shows the equal-courses as straight lines, the possibility of such a projection should be pointed out. By means of affine alteration, one can come closer to true-area in expanded areas, in that a prescribed finite segment of meridian in its entire length is brought into a fitting ratio to some central parallel of latitude through appropriate choosing of (m:n).

The last cylindrical prime-point map has no other outstanding properties (SNo. 97) and is only included for the sake of the table of terms.

Subbranch B: non-cellular (SNo. 98-152)

The non-cylindrical, although *cylinder-circular* projections, which constitute this subbranch, are usually called

improper-cylindrical projections [also called conventional-cylinder projections]. Their equations are: $x = F(\phi) = -F(-\phi)$, with the addition that the H may not be equal-interval parallels according to the equation $y = n\lambda$.

Order a: Prime-point Maps (SNo. 98,99)

This sort of *prime-point maps,* whose H must extend into infinity, have understandably never been proposed. Nevertheless, it should be pointed out that one can even accomplish area-true projections of this sort. To do so, one only need employ $y = \lambda \cdot g(\phi)$ and $x = F(\phi)$, where $F(\phi) = -F(-\phi)$ and $F\left(\pm\frac{\pi}{2}\right) = \infty$, and then determine the function $g(\phi)$ according to the stipulation for area-true $g(\phi) = \cos\phi : \frac{dx}{d\phi}$. If, for example, one lets $x = F(\phi) = \operatorname{tg}(\phi)$, then the result is $g(\phi) = \cos^3\phi$. This area-true cylinder-circular prime-map (SNo. 98), with the equations, $x = \operatorname{tg}\phi;\ y = \lambda\cos^3\phi$, or more generally, $x = n \operatorname{tg}\phi;\ y = \frac{\lambda}{n}\cos^3\phi$, is given in order to fill out the table of terms, as is the following projection, No. 99, which is not area-true, but reproduces all N true to scale ($x = n \operatorname{tg}\phi;\ y = \lambda\cos\phi$).

Order b, Class I: Ring-maps with equally divided central-prime-lines (SNo. 100-130)

Order b, the *ring-maps,* for which $F\left(\pm\frac{\pi}{2}\right)$ is finite, follows. A work by K.H. Wagner, 1932[x] gives valuable information about this group, which is being more and more recognized as significant for world maps. The following exposition is in complete agreement with the valuable paper mentioned above, but supplements it with systematic grouping and in a few other particulars.

Since these maps are *divided according to secondary circles,* the cells-- biangles on the globe between each two prime circles - are always reproduced area-similar; and they

are equal-area (on the whole, which does not necessarily mean true-area in the parts), if the entire surface of the world map is equivalent to the surface 4π of the uniform globe. The stipulation for this cell-equality reads: $\int_0^{\pi/2} \frac{y}{\lambda} \frac{dx}{d\phi} d\phi = 1$

The N $\pm\phi_w$ are true to scale, where $y = \lambda \cos \phi_m$; and true-shape exists on the central-H at the points $\pm\phi_w$, where $\cos \phi_w = \frac{dy}{d\lambda} : \frac{dx}{d\phi}$. If it is found in this manner that $\phi_w = 0$, then the whole basic-circle is considered true-shape, but only certain points of the other N ϕ_w where H and N intersect each other perpendicularly.

As main class I, we also obtain here those with homo-geneous central-H $(x = m\phi)$. For this class, the stipulation for cell-equality reads $m \int_0^{\pi/2} \frac{y}{\lambda} d\phi = 1$, and for true-sha[pe] $m \cos \phi_w = \frac{dy}{d\lambda}$.

Point-rings maps (SNo. 100-109)

In this subclass the prime-points are reproduced as points, that is $y = 0$ when $\phi = \pm\frac{\pi}{2}$. The *first group* shows nothing but scale-similar N according to the equation $y = n\lambda \cos \phi$, with an interval-true $(m = 1)$ and an only interval-similar $(m \gtrless 1)$ sort, plus the four types, No. 100-103, which are the cylindrical-circular border cases or the conical projections SNo. 59-62. For this group, the stipulation for cell-equality is $mn = 1$, the stipulation for scale-true N is $n = 1$, and $n = m$ for true-shape; then, true-shape exists for the whole central-H and for the whole basic-line. SNo. 100 is the already mentioned projection, proposed by Mercator and used much later by Sanson and Flamstead.

In a *second group*, not all the N are scale-similar; however, basic-line $(\phi = 0)$ and central-H $(\lambda = 0)$ should be in the same ratio m to their originals on the uniform globe, so that when $x = 0$, $y = m\lambda$ as when $y = 0, x \beta m \phi$.

In the first sort, *every H is a pair of straight* lines through the pole of the map and the dividing point on the basic-line, that is $y = m\lambda\ (\pi - 2\phi):\pi$. The types are differentiated only by the value of m. The stipulation of cell-equality results in:

$$m \int_0^{\pi/2} \frac{y}{\lambda}\, d\phi = m^2 \int_0^{\pi/2} \left(1 - \frac{2\phi}{\pi}\right) d\phi = m^2 \left[\phi - \frac{\phi^2}{\pi}\right]_0^{\pi/2} =$$

$$m^2 \frac{\pi}{4} = 1, \text{ thus } m = 2:\sqrt{\pi}$$

The N $\pm\phi_m$ are scale-true where $\cos\phi_m = m(\pi - 2\ \phi_m):$, and as far as true-shape is concerned, one finds from the equation $m \cos\phi_w = m\ (\pi - 2\ \phi_w):\pi\ \phi_w = 0$, thus, true-shape at the basic-line. SNo. 104 gives the general case, No. 105 gives the interval-true case, and No. 106, the equal-cell exception.*

In the second sort, every H is a semi-ellipse through the two poles of the map and the dividing point on the basic-line, thus $y = \frac{m\lambda}{\pi} \sqrt{\pi^2 - 4\phi^2}$. Now the stipulation for cell-equality becomes:

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* In all the projections with straight-line pairs as H, the single H cannot be represented with a uniform equation. Another equation is more valid for the hemisphere with negative ϕ than for the hemisphere with positive ϕ. To be exact, such doubly-symmetrical maps belong in our system in family V (c.f. SNo. 226); however, they are also listed in family II under the No. 104-106, 113-115, 121-126, 132 and 134. In all these cases we give the equation in the form valid for positive ϕ. Insofar as the projections are to be drawn as doubly-symmetrical, the same equations are also applicable for the hemisphere with negative ϕ; but the calculation does not employ the negative, but rather the positive values for x and ϕ, while x and ϕ negative are to be used in the drawing. One can, however, let the equations apply for whole world maps uniformly with the true signs for all greatnesses, but then one obtains projections which are no longer symmetrical to the basic-line, and thus belong to family IV of our system (c.f., SNo. 208-214).

$$m\int_0^{\pi 2}\frac{y}{\lambda}\,d\phi = m^2\int_0^{\pi 2}\sqrt{1-\frac{4\phi^2}{\pi^2}}\,d\phi = m^2\frac{\pi}{4}\left[\frac{2\phi}{\pi}\sqrt{1-\frac{4\phi^2}{\pi^2}} + \arcsin\frac{2\phi}{\pi}\right]_0^{\pi 2} =$$

$\frac{m^2\pi^2}{8} = 1$, thus $m = \sqrt{8}:\pi$

The $N \pm \phi_m$ are true to scale, where $\cos\phi_m = \frac{m}{\pi}\sqrt{\pi^2 - 4^2\phi_m}$; the basic-line is considered true-shape. SNo. 107 gives the general case, No. 108 gives the interval-true, and No. 109, the equal-cell exception. P. Apianus (Bienewitz) proposed No. 108 in 1524 as his projection no. II; it was later used by Arago in 1835.

Line-ring maps (SNo. 110-130)

In the first group of this subclass, the two polar-straight-lines are just as long as the central-H, thus $y = m\lambda : 2$ when $\phi = \pm\pi : 2$. The projections, mainly developed by Eckert, are combinations of the quadratic flat-map (SNo. 90) in the form $n = m$, thus $x = m\phi : y = m\lambda$, and one of each of the three preceding subclasses. Eckert[50] does not derive his projections as combination maps, but in single instances reproduces the globe first on the surface of a torus, from which he then projects onto a plane. Bludau[51] has correctly pointed out that one cannot "lay off" the surface of this torus onto a plane. Such a hybrid, the projection of the globe onto another likewise doubly curved surface, does not seem to me to serve much purpose. The projections of Eckert and those of O. Winkel fit quite simply as combination maps into our system, just as Winkel[52] himself proposed. He published a series of such maps which are distinguished by their agreeable appearance and an essential lessening of the distortion in the prime-forms. The equation valid for this group is $y = \frac{1}{2}(m + y_1)$, where the y of one of the preceding projections can be taken as y_1.

We analyse the group in sorts, according to which one of the three preceding sorts was referred to for the intermediary-projection of the quadratic flat map. In the case of the first sort, the first sort of the preceding subclass in the form n = m was employed, thus $y = \frac{m\lambda}{2}(1 + \cos\phi)$. The H are sine-lines, each through a dividing point on the two polar-straight-lines and the one on the basic-line. The cell-equality stipulation results in:

$$m\int_0^{\pi/2}\frac{y}{\lambda}\,d\phi = \frac{m^2}{2}\left[\phi + \sin\phi\right]_0^{\pi/2} = \frac{m^2}{2}\left(\frac{\pi}{2}+1\right) = 1,$$

thus $m = 2 : \sqrt{\pi + 2}$.

The N $\pm\phi_m$ are true to scale, where $\cos\phi_m = \frac{m}{2}(1 + \cos\phi_m)$

thus $\cos\phi_m = \frac{m}{2-m}$; and true-shape exists only on the basic-line

$\phi_w = 0$, since $\cos\phi_w = \frac{1}{2}(1 + \cos\phi_w)$, thus $\phi_w = 0$.

SNo. 110 gives the general case, No. 111 gives the interval-true, and No. 112 gives the equal-cell exception. No. 110 was developed in 1922 by Winkel and No. 112 was proposed in 1906 as projection no. V of Eckert.

When using the second sort (SNo. 104-106) in form n = m for intermediary reproduction, we obtain the equation $y = m\lambda\left[1 - \frac{\phi}{\pi}\right]$. The H are straight-line pairs, each through a dividing-point on the two polar straight lines and the point on the basic-line. The stipulation of cell-equality is:

$$m\int_0^{\pi/2}\frac{y}{\lambda}\,d\phi = \frac{m^2}{2}\left[\phi - \frac{\phi^2}{\pi}\right]_0^{\pi/2} = 3m^2\pi : 8 = 1, \text{ thus } m = \sqrt{\frac{8}{3\pi}}$$

The N $\pm\phi_m$ are scale true, where $\cos\phi_m = m(\pi - \phi_m):\pi$; and true-shape is valid for $\cos\phi_w = \frac{\pi - \phi_w}{\pi}$, which is fulfilled for the three points, $\lambda = 0$, $\phi_w = 0$ and $\phi_w = \pm 37°49.8'$.

SNo. 113 gives the general case, No. 114 gives the interval-true, and No. 115 gives the equal-cell exception. No. 114 is

the projection no. I of Eckert. The note on p.67 applies to the classification of this sort with straight-line pairs as H.

The last sort of the previous subclass serves in the intermediary-reproduction of the next sort (No. 116-118). Thus, $y = \frac{m\lambda}{2}\left[1 + \sqrt{1 - \frac{4\phi^2}{\pi^2}}\right]$. The H are *semi-ellipses* The stipulation for cell-equality reads:

$$m\int_0^{\pi/2} \frac{y}{\lambda}\, d\phi = \frac{m^2}{2}\left[\phi + \frac{\pi}{4}\left(\frac{2\phi}{\pi}\sqrt{1 - \frac{4\phi^2}{\pi^2}} + \arc\sin\frac{2\phi}{\pi}\right)\right]_0^{\pi/2} =$$

$\frac{m^2\pi}{16}(4 + \pi) = 1$, thus $m = 4 : \sqrt{\pi(4 + \pi)}$. True scale exists for $\pm\phi_m$, where $\cos\phi_m = \frac{m}{2}\left[1 + \frac{1}{\pi}\sqrt{\pi^2 - 4\phi_m{}^2}\right]$; and true-shape is rendered by $\cos\phi_w = \frac{1}{2}\left[1 + \frac{1}{\pi}\sqrt{\pi^2 - 4\phi w_2}\right]$, which is only possible for $\phi_w = 0$. SNo. 116 gives the general case, No. 117 gives the interval-true, and No. 118 gives the equal-cell exception which corresponds with Eckert's projection no. III.

Instead of *combination-maps* with the quadratic flat-map, the following group offers combination-maps with the *rectangular flat-map* SNo. 91, which corresponds with the equations $x = m\phi$; $y = n\lambda \cos\phi'$; (ϕ' their scale-true N). One can assign the equations $x = m\phi$; $y = \frac{1}{2}(n\lambda\cos\phi' + y_1)$ to such combination-maps, where y_1 is from the projection used for interposing. If, for the latter, one chooses a point-ring-map of the subclass No. 100-109, which produces $y_1 = 0$ for $\phi = \pm\frac{\pi}{2}$, then $y = \frac{n\lambda}{2}\cos\phi'$ for $\phi = \pm\frac{\pi}{2}$ fo for the combination-map. O. Winkel[52] recommended a projection of this form, in which a projection of the group, No. 100-103, with the equations $x = m\phi$; $y_1 = n\lambda$ was employed for the combination with the rectangular flat-map. Winkel employs here $m = n = 1$ (SNo. 120). The general form (SNo. 119) has the equations $x = m\phi$; $y = \frac{n\lambda}{2}(\cos\phi + \cos\phi')$. No. 121-123 are special cases of No. 119. If $n = m$ and $\phi' = 0$, this sort passes over into the earlier discussed

sort, No. 110-112. For the new sort (No. 119), the equal-area stipulation results as:

$$m\int_0^{\pi/2}\frac{y}{\lambda}\ d\phi = \frac{mn}{2}\int_0^{\pi/2}(\cos\ \phi + \cos\ \phi')\ d\phi = \frac{mn}{2}\left[1 + \frac{\pi}{2}\cos\ \phi'\right] = 1,$$

thus, $mn = 4 : (2 + \pi\cos\ \phi')$.

For scale-true N $\pm\phi_m$, one finds $\cos\ \phi_m = \frac{n}{2}(\cos\ \phi_m + \cos\ \phi')$, thus $\cos\ \phi_m = \frac{n}{2-n}\cos\ \phi'$, and for true-shape one finds

$m\cos\ \phi_w = \frac{n}{2}(\cos\ \phi_w + \cos\ \phi)$, thus $\cos\ \phi_w = n\cos\ \phi' : (2m - n)$. Of interest here is the special case (SNo. 122) $m = 1$ and $\phi' = 50°27.6'$, in which the stipulations both for being interval-true and cell-equal are fulfilled at the same time. Such an interval-true map by Winkel reproduces each cell between its two H area-equal, and the two N $\phi = \pm\ 50°27.6'$ true to scale, and, at the intersecting points of these N with the central-H, it is shape-true.

K.H. Wagner,[53] under the same assumption made by Winkel that $n = 1$, investigates the question concerning this projection: At which value $n_1 = \cos\ \phi'$ do the shortening of the basic-line and of the central-H become equal? But Wagner erroneously assumes as stipulation that $m = n_1$ for this case, and tries to find the solution by trial. Actually when one assumes that $n = 1$, the scale $y = \frac{\lambda}{2}(1 + n_1)$ applies at the basic-line, and the scale $x = m\phi$, on the central-H, for which the stipulation for cell-equality $m = 4:(2 + \pi n_1)$ applies. Regular shortening of the basic-line and of the central-H is required, then by $m = \frac{1 + n_1}{2}$; this results, in place of the value $\phi' = 31°20'20''$ as found by Wagner, in the quadratic equation for n_1, $4:(2 + \pi n_1) = (1 + n_1):2$, consequently $n_1 = 0.50426$; $m = 0.7529$ and $\phi' = 59°43.1'$. True-shape exists in this projection (No. 123) at the two points $y = 0$; $\phi = \pm\phi_w$, where $\cos\ \phi_w = \cos\ \phi' : (2m-1)$, that is, when the proper number value is inserted, $\phi_w = 65°50.14'$.

A further group of our subclass of line-ring-maps is distinguished by having *four scale-true* N $\pm\phi_1$ and $\pm\phi_2$. Such a form was used in 1482 by Donis and in 1849 by Donny; each of its H consists of a pair of straight lines, each passing through the dividing points on the two scale-true N of the same hemisphere. SNo. 124 gives the general form, No. 125 gives the interval-true, and No. 125, the equal-cell special case. The general equations are: $x:m\phi;$

$$y = \frac{\lambda}{\phi_2 - \phi_1} [(\phi - \phi_2) \cos \phi_1 - (\phi - \phi_1) \cos \phi_2]$$

The stipulation for cell-equality becomes:

$$m = \frac{2}{\pi} (\phi_2 - {}^2{}_1) : \left[\left(\frac{\pi}{4} - \phi_1\right) \cos \phi_2 - \left(\frac{\pi}{2} - \phi_2\right) \cos \phi_1\right].$$

Cell-equality and equidistance are not to be combined in this type of projection. See the note on p.67 concerning which family this group belongs to.

A generalization of the form $y = \frac{m\lambda}{2} (1 + \cos \phi) = m\lambda \cos^2 \frac{\phi}{2}$ can lead to another group of this subclass, if we, rather than using m in y, employ a different constant n of the m-value in $x = m\phi$, and other numbers for the numbers 2 of the right side. Such a strong generalization would be $x = m\phi;\ y = n\lambda \cos^c p\phi$, where $c = 1$ or $c = 2$ and p could be a true fraction. Actually a projection like this, with the form $c = 1$, $p =$, was proposed by K.H. Wagner,[54] where the constant n from the stipulation is determined for a scale-true N ϕ_m to $n = \cos \phi_m : \cos \frac{2\phi_m}{3}$. One can, at the same time, obtain equidistance by means of $m = 1$. Sno. 127 corresponds with the general form, No. 128 corresponds with the interval-true special case presented by Wagner, for which $n = 0.887$ and $\phi_m = 40°$. The stipulation for cell-equality for this group is:

$$\pi = \int_0^{\pi/2} y_\pi \frac{dx}{d\phi}\, d\phi = mn\pi \int_0^{\pi/2} \cos \frac{2\phi}{3}\, d\phi = \frac{3mn\pi}{2} \sin \frac{\pi}{3},$$

thus $mn = \frac{2}{3} \operatorname{cosec} \frac{\pi}{3} = 0.7698.$

SNo. 129 corresponds with this case. If one combines yet equidistance with cell-equality, employing thus $m = 1$, then $n = 0.7698$, and, true to scale, the N $\phi_m = 49°57'$, for which $\cos \phi_m : \cos \frac{2\ \phi m}{3} = 0.7698$.

Order b, Class II: Ring-maps with unequally divided central primary lines (SNo. 131-152)

Only one area-true point-ring-map (SNo. 102, with its special case SNO. 100) appears in class I with the equally central-H; and, rather than area-true line-ring-maps, area-equal ones are, at the most,possible. Class II, on the other hand, with the unequally divided central-H in its *subclass A*, offers a host of area-true projections. The stipulation for true-area here reads:

$\pi \sin \phi = \int_0^\phi y_\pi \frac{dx}{d\phi}\, d\phi$, where y_π is the value of y for $\lambda = \pi$.

The *first group* is distinguished in that y_π is given as function of x or ϕ through the prescribed border of the world map, while x as function of ϕ is prescribed in the second group.

If the border-H of the world map in the first group has the equation $y_\pi = F(x)$, then the equation for the H λ is $y = \frac{\lambda}{\pi} F(x)$, and the stipulation for area-true reproduction of the zone of the globe between $\phi = 0$ and $\phi = \phi$ reads
$\pi \sin \phi = \int_0^x F(x)\, dx$.

In the first sort we use the border of the world map of the cell-equal projections of the group (110-118) of the previous class, in other words, the sine-border of SNo. 112 for the first type (SNo. 131). Here, then $m = 2 : \sqrt{\pi + 2}$ and applies for the border of the world map; thus, the H $\lambda = \pi$, that equation $y = F(x)$, which one obtains from the equations

$x = m\phi$; $y = \frac{m\pi}{2}(1 + \cos\phi)$ of the H-line by means of elimination of ϕ. That results in:

$$F(x) = \frac{m\pi}{2}\left(1 + \cos\frac{x}{m}\right)$$

Thus, the stipulation of true-area results in:

$$\pi \sin\phi = \int_0^x \frac{m\pi}{2}\left(1 + \cos\frac{x}{m}\right) dx = \frac{m\pi}{2}\left(x + m\sin\frac{x}{m}\right)$$

And the equations of the projection No. 131 are:

$$\sin\phi = \frac{m}{2}\left(x + m\sin\frac{x}{m}\right);\ y = \frac{m\lambda}{2}\left(1 + \cos\frac{x}{m}\right) \quad \text{and } m = 2:\sqrt{\pi + 2},$$

from which the related values of x, ϕ and y can be calculated. Since the projection is area-true, one N ϕ_m being true to scale, according to the above derived stipulation $\cos\phi_m = \frac{y}{\lambda}$, also means true-shape at the point of intersection of this N with the central-H. One finds

$$\frac{y}{\lambda} = \frac{m}{2}\left(1 + \cos\frac{x}{m}\right) = \cos\phi_m$$

with the basic equation $\sin\phi_m = \frac{m}{2}\left(x + m\sin\frac{x}{m}\right)$ with elimination of ϕ_m for the calculation of x, this results in:

$$\frac{m^2}{4}\left(1 + \cos\frac{x}{m}\right)^2 + \frac{m^2}{4}\left(x + m\sin\frac{x}{m}\right)^2 = 1.$$

When the value $m = 2:\sqrt{\pi + 2}$, this equation is fulfilled for $x = 0.94315$, from which $\phi_m = 49°11'$ is calculated. This projection, also known as Eckert's projection no. VI, is thus area-true, scale-true at the N $\phi_m = \pm 49°11'$, and true-shape at the two points $\lambda = 0$; $\phi = \pm 49°11'$.

The second type (SNo. 132) obtains the border of the world map of the projection No. 115, which forms a rectilineal hexagon. We eliminate ϕ from the equations of the H-line for $\lambda = \pi$, $x = m\phi$; $y = m(\pi - \phi)$ and obtain $y = F(x) = m\pi - x$. The stipulation for true-area yields:

$\pi \sin\phi = \int_0^x (m\pi - x)\, dx = m\pi x \quad \frac{x^2}{2}$, thus $x = m\pi - \sqrt{m^2\pi^2 - 2\pi\sin\phi}$

as the single definitive equation of the projection No. 132, to which the second $y = \frac{m\lambda}{\pi} \left(\pi - \frac{x}{m}\right)$, thus $y = \frac{\lambda}{\pi} \sqrt{m^2\pi^2 - 2\pi \sin\phi}$ is added, and should be given according to No. 115 $m = \sqrt{8:(3\pi)}$. The stipulation $\cos\phi_m = y:\lambda$ for true-scale and true-shape is $\cos\phi_m = \sqrt{m^2 - \frac{2}{\pi}\sin\phi_m}$, from which one finds $\sin\phi_m =$ $\frac{1}{\pi} + \sqrt{1 + \frac{1}{\pi^2}} - m$ and $\phi_m = 55°10'$. This projection, known as Eckerts projection no. II, is thus area-true, scale-true at the N $\phi_m = 55°10'$, and shape-true at the two points $\lambda = 0$; $\phi = \pm 55°10'$. See the note on p.67 concerning which family it is related to.

The third type (SNo. 133) obtains the border-H of the projection No. 118 in the form of an ellipse. From the equations for the H for $\lambda = \pi$ $x = m\phi$; $y = \frac{m\pi}{2}\left(1 + \sqrt{1 - \frac{4\phi^2}{\pi^2}}\right)$ one finds by elimination from ϕ that $y = F(x) = \frac{m\phi}{2}\left(1 + \sqrt{1 - \frac{4x^2}{m^2\pi^2}}\right)$

so that the stipulation for true-area yields:

$$\pi \sin\phi = \int_0^x \frac{m\pi}{2}\left(1 + \sqrt{1 - \frac{4x^2}{m^2\pi^2}}\right) dx = \frac{m\pi x}{2} + \frac{m\pi x}{4}\sqrt{1 - \frac{4x^2}{m^2\pi^2}} +$$

$$+ \frac{m^2\pi^2}{8} \arcsin\frac{2x}{m\pi}$$ to which is added

$y = \frac{m\lambda}{2}\left(1 + \sqrt{1 - \frac{4x^2}{m^2\pi^2}}\right)$. If, in order to do away with the arc sin, one introduces $\sigma = \arcsin\frac{2x}{m\pi}$, thus $\frac{2x}{m\pi} = \sin\sigma$ and $\sqrt{1 - \frac{4x^2}{m^2\pi^2}} = \cos\sigma$, then one obtains the defining euqations for projection No. 133:

$(4 + \pi)\sin\phi = 4\sin\sigma + \sin 2\sigma + 2\sigma$; $y = 2\lambda\,(1 + \cos\sigma)$:

$\sqrt{\pi(4 + \pi)}$; $x = 2\pi\sin\sigma : \sqrt{\pi(4 + \pi)}$, if one has inserted the value

$\frac{m^2\pi}{4} = \frac{4}{4+\pi}$, that is the value for M from No. 115. The stipulation for true-scale and true-shape $\cos\phi_m = \frac{y}{\lambda} = \frac{m}{2}(1+\cos\sigma)$ which at the same time must be fulfilled with the general equation $(4+\pi)\sin\phi_m = 4\sin\sigma + \sin 2\sigma + 2\sigma$, yields through elimination of ϕ_m for the equation

$$(4+\pi)^2 = \frac{8(4+\pi)}{\pi}\cos^2\frac{\sigma}{2} + (4\sin\sigma + \sin 2\sigma + 2\sigma)^2,$$

which is fulfilled for $\sigma = 18°23'$; then, according to $(4+\pi)\sin\phi_m = 4\sin\sigma + \sin 2\sigma + 2\sigma$ one obtains $\phi_m = 40°29'$. This projection, known as Eckert's projection no. IV is area-true, scale-true at the N $\phi_m = \pm 40°29'$, and sha- -true at the two points $\lambda = 0$; $\phi = \pm 40°49'$.

The next sort uses the border of the world map of the equal-cell projections of the group, No. 104-109; SNo. 134 is related to No. 106. One must employ, then $m = 2:\sqrt{\pi}$ and $y_\pi = F(x) = m(\pi - 2\phi) = m(\pi - 2x)$, because there x equaled $m\phi$. The stipulation for true-area thus becomes:

$$\pi\sin\phi = \int_0^x (m\pi - 2x)\,dx = m\pi x - x^2$$

to which the equation $y = \frac{m\lambda}{\pi}\left(\pi - \frac{2x}{m}\right)$ is added. These determining equations for SNo. 134:

$$m = 2:\sqrt{\pi};\ \pi\sin\phi = m\pi x - x^2;\ y = \frac{m\lambda}{\pi}\left(\pi - \frac{2x}{m}\right)$$

can be transformed to

$$x = \sqrt{\pi}\left[1 - \sqrt{2}\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right];\ y = \lambda\sqrt{\frac{8}{\pi}}\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right).$$

The stipulation for true-scale and true-shape becomes:

$$y = \lambda\cos\phi_m = \lambda\sqrt{\frac{8}{\pi}}\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \text{ or } \sqrt{\pi} = \operatorname{cosec}\frac{\phi m}{2} - \sec\frac{\phi m}{2}$$

which gives $\phi_m = \pm 41°13.3'$. This area-true projection was proposed by Collignon in 1865. See the note on p. 67 concerning the family to which it belongs.

Type No. 135 uses the border of the world map of the equal-cell projection No. 109, thus $m = \sqrt{8}:\pi$ and $y_\pi = F(x) = m\sqrt{\pi^2 - 4\phi^2} = m\sqrt{\pi^2 - \frac{4x^2}{m^2}}$, sincex = $m\phi$ in that projection. The true-area stipulation, with the employment of $m = \sqrt{8}:\pi$, is

$$\pi \sin\phi = \sqrt{8}\int_0^x \left(1 - \frac{x^2}{2}\right)dx = \sqrt{2 - x^2} + 2 \text{ arc sin } \frac{x}{\sqrt{2}},$$

to which is added $y = \frac{2\lambda}{\pi}\sqrt{2 - x^2}$. If one introduces $\frac{x}{\sqrt{2}} = \sin\psi$ then $\pi \sin\phi = \sin 2\psi + 2\psi$; $x = \sqrt{2}\sin\psi$; $y = \frac{\sqrt{8}}{\pi}\cos\psi$.
The stipulation for true-scale and true-shape is $\cos\phi_m = \frac{\sqrt{8}}{\pi}\cos\psi_m$; from this, by singling out ϕ_m with the aid of the equation $\pi \sin\phi_m = \sin 2\psi_m + 2\psi_m$ for $_m$, one obtains the equation $\pi^2 = 4\psi_m(\psi_m + \sin 2\psi_m) + 4\cos^2\psi_m(3 - \cos^2\psi_m)$; this is fulfilled for $\psi_m = 32.7°$. Then, the equation

$$\sin\phi_m = \frac{\sin 2\psi_m + 2\psi_m}{\pi}$$ yields the value $\phi_m = 45°46'$.

This area-true projection No. 135, with elliptical border of the world map, was presented by Mollweide in 1805 and used in 1857 by Babinet.

Whereas the three Eckert Maps of the subclass were line-ring-maps, those by Collignon and Mollweide are point-ring-maps.

The following sort derives its world map border from the true-area projections No. 87 and 100; No. 136 forms the mean of y_π with *equal values for x*. This projection is the cylinder-circular border case of the conical-circular Nell Projection SNo. 70. $x = \phi$; $y = \lambda\cos\phi$, that is, the equations for the H $\lambda = \pi$; $y_\pi = \pi\cos x$, applies for No. 87. $x = \sin\phi$; $y = \lambda$, that is, the equation for the H $(\lambda = \pi)$ $y_\pi = \pi$, applies for No. 100. Thus, $y_\pi = F(x) = \frac{\pi}{2}(1 + \cos x)$ applies for the Nell Projection No. 136 (1890), so that the stipulation for true-area

is $\pi \sin \phi = \int_0^x \frac{\pi}{2} (1 + \cos x)\, dx = \frac{\pi}{2} (x + \sin x)$, to which the equation $y = \frac{\pi}{2} (1 + \cos x)$ is added. Only the basic-line $\phi = x = 0$ is scale-true here, and true-shape exists all along it.

The world map border of type no. 137 also stands as a form of reproduction between the projections No. 87 and 100, and in this type, as Hammer proposed it in 1900 in his incorrect understanding of Nell's idea. Hammer finds the mean of the y-values of projections No. 87 and 100, not using equal x-values, but *with equal values for* ϕ. In his projection he thus obtains

$y = \frac{\lambda}{2} (1 + \cos \phi)$ and for $\lambda = \pi$, $y_\pi = F(x) = \frac{\pi}{2} (1 + \cos \phi)$.

The true-area stipulation, before integrating, reads $2\pi \cos\phi \, d\phi = 2F(x)\, dx = \pi(1 + \cos \phi)\, dx$; thus, as an integral equation,

$x = \int_0^\phi \frac{2 \cos\phi}{1 + \cos\phi} d\phi$, which, when integrated, results in

$$x = 2\phi - \operatorname{tg} \frac{\phi}{2} .$$

To this belongs the second determining equation of the Hammer Projection: $y = \frac{\lambda}{2} (1 + \cos \phi)$. This area-true projection, also, is scale-true only on the basic-line $\phi = 0$, and true-shape exists all along it. Both projections are most differentiated with high values for ϕ. For example, when $\lambda = 90°$ and

	$\phi = 45°$	$\phi = 90°$
for SNo. 136	$y = 0.2732$; $x = 0.7400$	$y = 2.278$; $x = 1.106$
for SNo. 137	$y = 0.2681$; $x = 0.7424$	$y = 1.571$; $x = 1.142$

The next sort (No. 138, 139) takes $F(x)$ from group No. 119-123, employing $F(x) = \frac{n\pi}{2} \left(\cos \frac{x}{m} + \cos \phi'\right)$, where $mn = 4:(2 + \pi \cos \phi')$. According to No. 119 one finds

$$\pi \sin \phi = \int_0^x \frac{n\pi}{2} \left(\cos \frac{x}{m} = \cos \phi'\right) dx = \frac{n\pi}{2} \left[m \sin \frac{x}{m} + x \cos \phi'\right]$$

as the stipulation for true-area, so that the equations for

projection No. 138 read:

$$m \sin\frac{x}{m} + x\cos\phi' = \frac{2}{n}\sin\phi;\; y = \frac{n\lambda}{2}\left(\cos\frac{x}{m} + \cos\phi'\right);$$

$mn = 4:(2 + \cos\phi')$. The stipulation for true-scale and true-shape is $\cos\phi_w = \cos\phi_m = \frac{n}{2}\left(\cos\frac{x}{m} + \cos\phi'\right)$; this, together with the basic-equation $\sin\phi_m = \frac{n}{2}\left(m\sin\frac{x}{m} + x\cos\phi'\right)$ permits the singling out of ϕ and thereby the calculation of x from the fixed values m, n, $\cos\phi'$, so that ϕ_m also results from the equation for $\sin\phi_m$. The following statements regarding SNo. 139 may serve as example of such a projection - which K.H. Wagner[53] also briefly mentioned. $\cos\phi' = \frac{2}{\pi}$, thus $\phi' = 50°27{,}6'$ might be employed, so that according to the equation $mn = 4:(2 + \pi\cos\phi') = 1$. If one then requires the central-H to be half as long as the basic-line, then one finds, from the stipulation $x\left(\phi = \frac{\pi}{2}\right) = y\left(\phi = 0; \lambda = \frac{\pi}{2}\right)$, the equation $\sin\frac{\pi+2}{4m^2\,\pi} = 2 - \frac{\pi+2}{2m^2\,\pi}$, which results in m = 0.0-46, thus $n = \frac{1}{n} = 1.1054$. The length of the central-H and of the half basic-line becomes 2.8426 and that of the entire pole-line = 1.5944. The stipulation for true-scale and true-shape is

$$\sin\frac{x}{m} + \frac{x\cos\phi'}{m} = \frac{1}{m}\sqrt{4m^2 - \left(\cos\phi' + \cos\frac{x}{m}\right)^2}$$ which is very nearly fulfilled for x = 1, and the basic-equation

$\sin\phi = \frac{1}{2}\sin\frac{x}{m} + \frac{n}{\pi}x$ for ϕ. Such a true-area map is thus scale-true on the N $\phi_m = \pm 53°0.5'$; and true-shape at the two points $\lambda = 0$; $\phi_w = \pm 53°0.5'$.

The following sort (SNo. 140-142) is the transformation of group No. 127-130 from cell-equality (No. 120) to true area. K.H. Wagner[55] also gave a special case of this sort. The stipulation for true-area

$\pi \sin = \int_0^x n\pi \cos \frac{2x}{3m} dx = \frac{3n\ m}{2} \sin \frac{2x}{3m}$ follows from the equations $x = m\phi;\ y = n\lambda \cos \frac{2\phi}{3}$, where $mn = 0.7698$ is employed. The equations for projection No. 140 are, then:

$$\sin \frac{2x}{3m} = \frac{2}{3mn} \sin \phi; \quad y = n\lambda \cos \frac{2x}{3m} = \frac{\lambda}{3m} \sqrt{9m^2n^2 - 4 \sin^2\phi}$$ and

when employing $mn = 0.7698$,

$$\sin \frac{2x}{3m} = 0.8660 \sin \phi; \quad y = \frac{n\lambda}{2} \sqrt{4 - 3 \sin^2\phi} = \frac{n\lambda}{2} \sqrt{1 + 3 \cos^2\phi}.$$

One finds for true-scale and true-shape:

$\cos \phi_m = \frac{n}{2} \sqrt{1 + 3 \cos^2\phi_m}$, thus $\cos \phi_m = n : \sqrt{4 - 3n^2}$. Which N is to be scale-true depends only on n. Wagner employs $n = m = \sqrt{0.7698}$ for his projection (SNo. 141), which makes the central-H half as long as the basic-line, although this is not necessary. With his assumption, the N $\phi_m = \pm 53°41,8'$ become scale-true. In such a projection, one could also make every other N-pair $\pm\phi_m$ scale-true by using $n = 2 \cos \phi_m : \sqrt{1 + 3 \cos^2\phi_m}$. For example, for $n = 1$ (SNo. 142) the basic-line $\phi = 0$ would be true to scale, and all along it there would be true-shape; with another choice of n, the n would of course still be scale-true, but only two points would be true-shape.

In the last sort of the group, every border meridian $\lambda = \pm\pi$ of the world map is a parabola. In the projection proposed by Craster[56] in 1929 (SNo. 143), every central-H is made half as long as the basic-line 2a. Thus, entire true-area requires $a = \sqrt{3\pi}$. All H are parabolae of the equation $y = \lambda \sqrt{\frac{3}{\pi}} \left(1 - \frac{4x^2}{3\pi}\right)$ and the equation $x = \Phi = 3\pi \sin \frac{\phi}{3}$ corresponds with each N. One recognizes that the projection is area-true, since every globe biangle K between the H $\lambda = 0$ and $\lambda = \lambda$, as well as every globe zone Z between $\phi = 0$ and $\phi = \phi$ are reproduced area-true. On the uniform-globe K has the area 2λ and on the map:

$$2 \int_{x=0}^{x=a/2} y\,dx = 2\lambda\left[\sqrt{\frac{3}{\pi}}\,x - \frac{4x^3}{(3\pi)^{3/2}}\right]_0^{\frac{1}{2}\sqrt{3\pi}} = 2\lambda\left[\frac{3}{2} - \frac{1}{2}\right] = 2\lambda \text{ as on the globe.}$$

Zone Z on the globe has an area of $2\pi \sin\phi$ and on the map,

$$2 \int_{x=0}^{x=\phi} y(\lambda=\pi)\,dx = 2\left[x\sqrt{3\pi} - \frac{4x^3}{\sqrt{27\pi}}\right]_0^{\phi\sqrt{3\pi}\sin\frac{\phi}{3}} = 2\pi\left[3 \sin\frac{\phi}{3} - 4 \sin^3\frac{\phi}{3}\right]$$

thus, again, just as on the globe, since $\sin\phi = 3 \sin\frac{\phi}{3} - 4 \sin^3\frac{\phi}{3}$ can be generally applied. If one writes the equations of every H-parabola λ in the form $x^2 = \frac{p}{2}(\Lambda - y)$ where Λ is the distance of the vertex of the parabola from its source $x = y = 0$, then $\Lambda = \lambda\sqrt{\frac{3}{\pi}}$ and the parameter, $p = \frac{\pi\sqrt{3\pi}}{2\lambda}$. True-scale exists when $\cos\phi_m = \sqrt{\frac{3}{\pi}}\left(1 - \frac{4}{3}\sin^2\phi_m\right)$ which occurs for $\phi_m = \pm 36°45'$. Consequently, true-shape exists at the two points $\lambda = 0$; $\phi = \pm 36°45'$.

The world map border in the following group of area-true projections is determined, when the interval-law for the N is *prescribed* as $x = \Phi = f(\phi)$. In order to have true-area,

$y\frac{dx}{d\phi} = \frac{d(\lambda \sin\phi)}{d\phi} = \lambda\cos\phi$. In the only example of this family, thus far, the distribution $f(\phi)$ is of equal ratio to the radius-law of the all-circular projections (SNo. 2), that is $x = m \operatorname{tg}\frac{\phi}{2}$.

$y = \frac{2\lambda}{m}\cos\phi\cos^2\frac{\phi}{2}$ results from $y = \lambda\cos\phi : \frac{dx}{d\phi}$. The N $\pm\phi_m$ are scale-true here, where $\cos\phi_m = \frac{2}{m}\cos\phi_m \cdot \cos^2\frac{\phi m}{2}$, thus $\cos^2\frac{\phi m}{2} = \frac{m}{2}$. SNo. 144 gives the general case, No. 145 the special case with scale-true basic-line ($m = 2$), and No. 146 gives the instance in which the central-H are half as long as the basic-line. This is fulfilled for $m = \sqrt{\pi}$, where true-scale for $\cos^2\frac{\phi m}{2} = \frac{\sqrt{\pi}}{2}$, i.e. for $\phi_m = \pm 35°32.5'$, and true-shape at the two points $\lambda = 0$; $\phi_w = \pm 35°32.5'$. The projection of Prepetit-Foucaut was proposed in this form in 1862.

In a *last group* of our area-true subclass, we compile *affine* transformations of their group (No. 131 - 143), in which we increase all y at the same ratio, as which we decrease all x. It is clear that true-area is maintained. Naturally, one only undertakes such transformation in order to obtain some advantage. For example, it can be appropriate to achieve true-scale and true-shape for the entire basic-line, whereas the original projection has true-scale on the two N $\pm\phi_m$, although it only exhibits true-shape at two points. It is in this manner that our projection No. 147 arises from the Eckert Projection no. VI with sine-lines, when the y are reproduced with $\frac{\sqrt{\pi + 2}}{2} = 1.1338$ and all the x are divided by the same number; the same applies for our No. 148 from the Eckert Projection no. IV (SNo. 138) with H-ellipses with the aid of the factor $\frac{1}{4}\sqrt{\pi(4 + \pi)} = 1.1842$, and finally No. 149 from the Mollweid Projection No. 135 with the aid of the factor $\frac{\pi}{\sqrt{8}} = 1.1107$. Bourdin has calculated tables for the equation common to SNo. 149 and 135, $\pi \sin\phi = \sin 2\psi + 2\psi$, from which tables the ψ corresponding with each ϕ-value can be found. The type of curve of the H as sine-lines in No. 131 and 147, and all ellipses in projections No. 133, 135, 143 and 149, is maintained. The angle defects decrease sharply in the equatorial zone because of the affine transformation, but increase in the pole areas.

The two area-true cylindrical projections by Lambert (SNo. 87) and Behrmann (SNo. 89) can be considered as affine transformations of each other. Also affine to each other are the flat maps (No. 90 and 91), the Mercator maps on intersecting- and contiguous-cones (No. 94 and 95), and the four projections of the group No. 100-103.

In the last subclass of this subbranch the projections are not area-true, but are only *area-similar*, since all N are scale-similar, according to the equation $y = n\lambda \cos \phi$. They are consequently point-ring-maps. In the group of this subclass discussed thus far, the world map border is an ellipse, of which the axes, central-H and the basic-line are at a ratio of 1:2 to each other.

Thus, for $\phi = \pm\frac{\pi}{2}$, $x = \pm n\frac{\pi}{2}$. The simple interval rule $x = n\frac{\pi}{2}\sin\phi$ applies for No. 150-152, thus, in equal ratio to the radius rule of the orthographic projection (SNo. 3). The stipulation for true-shape $\cos\phi_w = \frac{dy}{d\lambda} : \frac{dx}{d\phi}$ becomes $\cos\phi_w = n\cos\phi_w : \left(\frac{n\pi}{2}\cos\phi_w\right) = 2:\pi$, i.e. the two points $\lambda = 0$; $\phi = \pm 50°27.6'$ are always shape-true in all the projections of this sort (different values n). These are the same shape-true points as in the Winkel Projection (SNo. 122). SNo. 150, with n = 1, gives the special case with true-scale of all N; No. 151 is that case in which all cells are area-true (n = 8:); and No. 152 gives the special case in which the H $\lambda = \pm\frac{\pi}{2}$ are circular, which applies for $n = 1:\sqrt{\pi}$. The projection in this latter form was proposed by Fournier in 1646 as his projection no. II.

Family II, Branch B: Straight Symmetrical, not divided according to secondary circles (SNo. 153-157)

There follow yet the straight-symmetrical projections which are not divided according to secondary circles. Such projections have been proposed for quite a long time. The projections SNo. 153-155 are point-ring-maps only for $\lambda^2 < \frac{\pi^2}{4}$, but are line-ring-maps for $\lambda^2 > \frac{\pi^2}{4}$, while No. 156 remains a point-ring-map up to $\lambda^2 = \pi^2$. No. 153 is the cylinder-circular border case of SNo. 82 under the special assumption $\delta_1 = 0$, $\delta_2 = \pi$, $\delta_3 = \frac{\pi}{2}$, thus $\phi_1 = \frac{\pi}{2}$, $\phi_2 = -\frac{\pi}{2}$; $\phi_3 = 0$. The projection is interval-true, with parallel N-straight lines ($x = \phi$) and circular H through the two polar points $\lambda = 0$; $= \pm\frac{\pi}{2}$ and the dividing-point $y = \lambda$ on the basic-line $\phi = 0$. That means that the equation for y is

$y^2 + \frac{y}{\lambda}\left[\frac{\pi^2}{4} - \lambda^2\right] = \frac{\pi^2}{4} - \phi^2$. According to this rule, only the hemisphere between the H $\lambda = -\frac{\pi}{2}$ and $\lambda = +\frac{\pi}{2}$ can be drawn with a full circle. The equation $y = \sqrt{\frac{\pi^2}{4} - \phi^2} + \lambda - \frac{\pi}{2}$ applies for the H $\lambda^2 > \frac{\pi^2}{4}$. The H are semicircles with radius $\frac{\pi}{2}$, each through

the dividing-point on the basic-line. The projection was presented in 1524 by P. Apianus (Bienewitz) as projection no. I. In the drawing which he produced (SNo. 154), the scale on the basic-line is only two-thirds that on the central-H, so that the full-circle with the radius $\frac{\pi}{2}$ reaches from $\lambda = -\frac{3\pi}{4}$ to $\lambda = +\frac{3\pi}{4}$ and the equations are $x = \phi$ and

$$\text{for } \lambda^2 < 9\pi^2 : 16 \quad y^2 + \frac{y}{\lambda}\left|\frac{3\pi^2}{8} - \frac{2}{3}\lambda^2\right| = \pi^2 - 4\phi^2 \quad \text{and}$$

$$\text{for } \lambda^2 > 9\pi^2 : 16 \quad y + \sqrt{\frac{\pi^2}{4} - \phi^2} + \frac{2\lambda}{3} - \frac{\pi}{2}.$$

The statement,[57] that every arc on the Apian map is not only extended to the straight line $x = \pm\frac{\pi}{2}$, but to the H-arc of the next smaller λ-value, is senseless with respect to the projection rule. As a picture of each prime-point, this results in a series of unconnected points, the number and inclination of which depends on the interval chosen by chance for the H drawn; and it changes into straight lines $x = \pm\frac{\pi}{2}$ with the drawing of the H, which one would therefore have to accept as pictures of the prime-points. Projection No. 153 is scale-true and shape-true on the basic-line.

SNo. 155, proposed in 1527 by Glareanus (Loritz), has the same H as No. 153, but the interval rule of the N-straight lines is not $x = \phi$, but $x = \sin\phi$, as on the area-true cylindrical projection (No. 87), by Lambert. The equations, then, are $x = \sin\phi$

$$\text{and for } \lambda^2 < \frac{\pi^2}{4} \quad y^2 + \frac{y}{\lambda}\left|\frac{\pi^2}{4} - \lambda^2\right| = \frac{\pi^2}{4} - \sin^2\phi \quad \text{and for}$$

$$\lambda^2 > \frac{\pi^2}{4} \quad y = \sqrt{\frac{\pi^2}{4} - \sin^2\phi} + \lambda - \frac{\pi}{2}.$$

This projection, too, is scale-true and shape-true on the basic-line.

This applies also for No. 156, a cylinder-circular projection, which reproduces not the hemisphere, but the full sphere in one circle; it was proposed as projection no. III by van der Grinten in

1904. On its copy in Zoppritz-Bludau[58] the latitude figures should be changed to 85° and 89° (printing error!). The map enlarges the polar area strongly in a meridianal direction. The true-shape on the basic-line ($\phi = 0$) makes it evident that

$\frac{dx}{d\phi} = \frac{dx}{d\mu} \cdot \frac{d\mu}{d\phi} = \sec \mu \cdot \sec^2 \frac{\mu}{2}$, thus assuming the value 1 for

$\phi = \mu = 0$, as follows from the projection's equation

$x = \pi \operatorname{tg} \frac{\mu}{2}; \ \sin \mu = 2\phi : \pi.$

As was already mentioned concerning SNo. 78, p.54, being cylinder-circular can be combined with being counter-azimuthal. If the counter-azimuthal point is $\lambda = 0$; $\phi = \phi_0$, and the coordinate break on the map is the point $\lambda = 0$; $\phi = \phi_0$, then the stipulation for being counter-azimuthal reads $\cot A = x:y$, where A must be equal to $(\cos \phi_0 \operatorname{tg} \phi = \sin \phi_0 \operatorname{tg} \lambda) : \sin \lambda$ [A = Azimuth].

As stipulation for being cylinder-circular, we join with this $x = F(\phi)$. The projection is symmetrical to the basic-line only when $\phi_0 = 0$, that is, when the counter-azimuthal aiming-point lies on the equator. If one now employs $F(\phi) = (\phi_0 - \phi)$, then one has a circle-interval-true counter-azimuthal projection, in which the interval are based on the latitudes. If $\phi_0 >< 0$, then the projection belongs in family IV, since there is no symmetry to the basic-line. In this manner we obtain found cylindrical projections:

counter-azimuthal, doubly symmetrical ($\phi_0 = 0$), circle-interval-true in family II
counter-azimuthal, doubly symmetrical ($\phi_0 = 0$), not interval-true in family II
counter-azimuthal not doubly symmetrical ($\phi_0 >< 0$) circle-interval-true in family IV
counter-azimuthal not doubly symmetrical ($\phi_0 >< 0$) not interval-true in family IV

Only the latter two are listed as the general cases in family IV of the system under SNo. 215 and 216; the special cases ($\phi_0 = 0$) would belong, however, to family II, branch B.

For the sake of being complete, we include as last projection (SNo. 157) of family II a cylinder-circular prime-point map not divided according to secondary circles; one obtains basic-line, when one introduces a non-homogeneous coordinate in the place of the homogeneous coordinate $y = \lambda \cos \phi$ - for example, $y = m \sin \frac{\lambda}{3} \cos \phi$ - and the other coordinate $x = n \operatorname{tg} \phi$ remains unchanged.

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FAMILY III, BRANCH A: CURVED SYMMETRICAL, ROW-CIRCULAR (SNo. 158-181)

The third family has the double-symmetry to the central-H and to the basic-line in common with the second family, but it does not have central-circularity. Its N are same-centered circles, as in families I and II, where the special case of parallel straight lines comes into consideration. In branch A of family III all the N are yet circles, but not circles of one center; here the centers lie on a straight line. I call such projections row-circular. According to Tissot-Hammer[59] [and according to Bourgeois Furtwangler[60]], one would assume that the term *"polyconical"* also applies, since one reads there (in T.H.): "'Polyconical projections.' should refer to those projections in which the parallels are represented by circles, the centers of which lie in a straight line." But then the authors do not stick to their definition; instead, they treat the orthogonal and oblique circle-grids, where the N are circles as well, the centers of which lie on a straight line, in two other sections of their work which are coordinated with the section entitled "Polyconical Projections." It seems that the concept that the *meridians may not be circles* is attached to the term "polyconical," but this is not made clear to the reader. It is evident that all these projections should be comprised

in one group, *row-circular* projections, for which Hammer's definition of polyconical applies literally, even though Hammer himself actually means only one subdivision of the thus defined projections. Which of those subdivisions should be called polyconical can best be determined according to the projections thus designated originally. In the case of these particular projections, one imagined a contiguous cone laid on each circle of latitude on the uniform globe, and considered every circle of latitude on the surface of the map as an arc with its cone-lateral as radius. Such "polyconical" projections obtain a straight-line, scale-true central-H, on which the centers of the N lie; and every N (ϕ) has as its radius its cone-lateral $r = \cot \phi$. This is the original use of the word "polyconical," to which one should also adhere in following definitions. Unfortunately, we find systematic vaguenesses in Gretschel and Herz, too. Gretschel[61] says, in his introductory definition of the polyconical projections: "The parallels appear everywhere in their true inclination," but in the same section of his work he handles the "rectangular-polyconical" projections, to which this does not apply. In Herz[62], § 30, "Polyconical Projections," polyconical is the same as row-circular, since the Lambert-Lagrange true-shape circle-grid (SNo. 169 - 173), projection 1 by Fournier (No. 175), the globular projection (No. 158), and

even the Mercator-Bonne Projection (No. 59) are designated as polyconical. Since the N of the latter named projection are same-centered, Herz could just as well have called all true conical projections polyconical. It is most surprising, however, that we find the section entitled "Polyconical Projections" in Chapter II, on conical projections, of which his definition reads: "The surface of the cone is always considered as a circle-cone, the axis of which is shared with the axis of the earth. The meridians must then necessarily appear as straight lines, which extend from the tip of the cone; the parallel circles as circles, the center point of which is the tip of the cone. The border cases, where the cone becomes a plane or a cylinder, are included. Thus, by conical projections Herz means the general conical projections, which must be conical-circular as well as conical-radial [in the border cases, radial-circular and radial, and/or cylindrical-circular and cylindrical-radial]; in footnotes, he then excludes expressly only cylindrical-circular, but not cylindrical-radial projections such as the one by Mercator (Sanson) (SNo. 95) and the one by Donny (No. 125).[64] But then he handles, in the same Chapter II, the row-circular projections, on which neither the N nor the H correspond with the definition he gives.

Concerning contradictions, such as the above, which have sneaked into the study of maps because of careless definition, I prefer only the designation "row-circular" for the above discussed comprehensive group; and for the subdivision, which corresponds with the original use of the word "polyconical," I prefer the designation *"all-circular"* for which the foreign word "panconical" would be said. In fact, the word "all-conical (panconical)" makes more sense than many-conical (polyconical), since not only *many* but *all* cones from the plane to the cylinder come into consideration. Thus, the N of *row-circular* projections are circles, the centers of which do not fall on one point, but on a straight line; and for the subdivision of *all-conical* (panconical) projections, the stipulations must be added that the straight central H is scale-true and the radius of each N is equal to the lateral of that cone which is contiguous with the globe at the original of the N concerned.

Each N of a row-circular projection is determined when the following are given for it as function of ϕ (c.f., Illustration 5, plate I): the radius SM = SB = r and the interval OB = Φ of its intersecting point B with the central-H, measured from the center of the map O. The distance OS of the central point

S from the center of the map is to be called s. And the orientation of the point M, the picture of the global point (ϕ/λ) on the N-line, is determined by means of the angle OSM, which is dependent on ϕ and λ and could be called β. The equation for the N themselves is, then, $y^2 + (\Phi + r - x)^2 = r^2$, where, considering the double-symmetry

$$\Phi(-\phi) = -\Phi(\phi),\quad s(-\phi) = -s(\phi),\quad \beta(-\lambda) = -\beta(\lambda)$$

and for

$$\phi = 0 \qquad \Phi = \frac{1}{r} = 0$$

Projections with circular primary-lines
and scale-true central-primary-line

(SNo. 158 - 165)

In the first branch, the row-circular projections, the N never intersect themselves. Φ increases as ϕ increases and r decreases in such a manner that $\Phi_2 - \Phi_1 < 2(r_1 - r_2)$ must always be for $\phi_2 > \phi_1$. Thus far, only polar-point-maps (order a) have been proposed in this branch, in which $\Phi = a$ for $\phi = \pm \frac{\pi}{2} \bar{r} = 0$, the central-H thus showing the length 2a. In the regular class of this order, all H are circles through the two prime-points of the map PP_1 (Illustration 5). These projections are called *circle grids of the first class.* Each λ is determined in them when the following are given as functions of λ: their radius $TP = TL = \rho$ and the distance

OL $= \lambda$ of the point of intersection L with the basic-line, measured from the center of the map O. The equation for the H is, then: $x^2 + (y + \rho - \Lambda)^2 = \rho^2$, where $\Lambda = \frac{1}{\rho} = 0$ for $\lambda = 0$. $a = a : \rho$ applies for the angle a between the H $\lambda = 0$ and λ.

In the *first subclass A*, we accept the central-H as *scale-true;* thus,

$$\Phi = \phi; \quad a = \frac{\pi}{2} \text{ and } \rho = \left(\frac{\pi^2}{4} + \Lambda^2\right) : (2\Lambda).$$

In group A the basic-line is also scale-true, thus,

$$\Lambda = \lambda \quad \text{and} \quad \rho = \left(\frac{\pi^2}{4} + \lambda^2\right) : (2\lambda)$$

The sorts vary according to the radius rule for r. In sort a, one point each on two II-circles $\pm\lambda_0$ is prescribed for each N ϕ except for the point $(y = 0 ; x = \phi)$. For this, SNo. 158 provides for the semi-circles $\lambda_0 = \pm \frac{\pi}{2}$ equally spaced. That is the well-known *globular projection* by Nicolosi (1660), which is generally used only for representations of hemispheres and is quite simple to draw, although the formulae for r and ρ look quite complex:

$$2r = \left(\frac{\pi^2}{4} + \phi^2 - \pi\phi\sin\phi\right) : \left(\frac{\pi}{2}\sin\phi - \phi\right);$$

$$2\rho = \left(\frac{\pi^2}{4} + \lambda^2 - \pi\lambda\sin\lambda\right) : \left(\frac{\pi}{2}\sin\lambda - \lambda\right)$$

The projection is also quite incorrectly named after Arrowsmith, who used it 134 years after Nicolosi.

If one stipulates the points for determining the N on the equal-spaced H, $\lambda = \pm\pi$ of the full globe picture, rather than on the H, $\lambda = \pm\frac{\pi}{2}$, then one obtains the *full globular projection* (SNo. 159) which I pointed out in 1922[65].

$$OL = \pi; \quad OT = -\frac{3\pi}{8}; \quad \rho = \frac{5\pi}{8}$$

applies for its border meridian; and

$$x = \frac{5\pi}{8} \sin \phi; \quad y = \pm\frac{\pi}{8} (3 + 5 \cos \phi)$$

applies for the dividing point ϕ on it. Similar to this projection is the apparent globular projection (SNo. 160), in which the dividing points of the N ϕ on the world map border are the points

$$x = \frac{5\pi}{8} \sin \frac{3\phi}{4}; \quad y = \pm \frac{\pi}{8} \left(3 + 5 \cos \frac{3\phi}{4}\right)$$

Van der Grintens' projection (SNo. 160) also places the dividing points of the N ϕ on the border meridian of the full globe; the equation for this in projections No. 159 - 161 reads

$$x^2 + \left(y - \frac{3\pi}{8}\right)^2 = \left(\frac{5\pi}{8}\right)^2$$

with van der Grinten, the dividing point lies at times on the straight line $\quad y + \frac{\pi}{4} = x\sqrt{10}\,(\pi - \phi):\phi$

For sort b, the all-conical (panconical) radius rule $r = \cot\phi$ applies; and, since all H have already been determined by means of the formula for the group, the only one of this sort is SNo. 162, the all-conical circle-grid with scale-true basic-line.

The radius rule for the last sort of this family is determined by means of the stipulation that the *circle-grid* should be *orthogonal*. The general stipulation,

$$\Phi^2 + \Lambda^2 = 2(\rho\Lambda - r\Phi)$$

yields, together with the other stipulations of the group,

$$\Phi = \phi; \qquad \Lambda = \Lambda \quad \text{and}$$

$$\rho = \left(\frac{\pi^2}{4} + \lambda^2\right) : (2\lambda) \qquad \text{for } r = \left(\frac{\pi^2}{4} - \phi^2\right) : (2\phi)$$

with which the type No. 163 is determined.

Equal-cellularity is stipulated in the second group B, which does without a scale-true basic-line. This results for the radii of the H in $\rho = \pi : (2 \sin \alpha)$, where α is the representation of the angle λ at the poles (Illustration 6, table III) and the equation

$\alpha = \frac{8\lambda}{\pi} \sin^2 \alpha + \frac{1}{2} \sin 2\alpha$ suffices, so that the sector $OL = \Lambda = \frac{\pi}{8} tg \frac{\alpha}{2}$. In sort a of this group the N are determined by having two other H $\pm\lambda_0$ aside from the central H, equal spaced. If $\alpha=\alpha_0$ for this, and $\rho = \rho_0$, then for the dividing points on them we obtain the coordinates for ϕ:

$$x_0(\phi) = \rho_2 \sin(2\phi\ \alpha_0:\pi); \quad y_0(\phi) = \pm\rho_0[\cos\ (2\phi\ \alpha_0:\pi) - \cos \alpha_0].$$

The cell-equal type (No. 164) applies for $\lambda_0=\pi$ and has, therefore, scale-true central H and equally spaced world border meridian. It does not vary much from true-area. The following little table gives the values α, ϕ and Λ for λ of 15° to 15°, and the values x_0 and y_0 on the world border $\lambda = \pm\pi$ for ϕ.

Cell-equal circle-grid with scale-true central H and equal spaced world border (No. 164)

[See above noted table on page 95a following.]

Illustration 6 gives the picture of this circle-grid corresponding to a globe with a radius of (100 : 3) mm.

In a second sort b, one could stipulate for the determining of the N that the zones between the N also be area-equal. The only type of this sort would be No. 165, the cell- and zone-equal circle-grid. For this, one would have to use the value $\rho_0=$ 1.673 given from No. 164, and α_0 = 110°7' for the intersection point of the Nϕ on the world map border $\lambda = \pi$, in order to find the angle α, which determines the coordinates of this intersection

TEXT NOT REPRODUCIBLE

Fächergleiches Kreisnetz mit maßtreuer Mittel-H und gleichteiligen Weltgrenzen

(Nr. 16[illegible])

Fußl-Länge λ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
σ	0°	15° 0'	31° 45'	49° 15'	61° 37'	71° 52'	80° 22'	87° 23'	93° 25'	98° 31'	102° 57'	10[illegible]° [illegible]'	110° 7'
ρ	∞	5,020	2,75[illegible]	2,013	1,[illegible]	1,[illegible]	1,[illegible]	1,572	1,5[illegible]	1,5[illegible]	1,612	1,[illegible]	1,673
Δ	0	0,[illegible]	0,[illegible]10	0,7[illegible]	0,[illegible]	1,1[illegible]	1,3[illegible]	1,5032	1,6[illegible]	1,8250	1,9[illegible]	2,1[illegible]	2,2152

Für λ … φ	0°	15°	30°	45°	60°	75°	90°
x_0	0	0,5257	1,0[illegible]	1,[illegible]	1,[illegible]	1,[illegible]	1,570[illegible]
y_0	2,2[illegible]	2,1[illegible]	1,[illegible]	1,[illegible]	1,0[illegible]	0,[illegible]	0

point (1) $x_o = \rho_o \sin\alpha$; $y_o = \rho_{\hat{\theta}} (\cos\alpha + \cos\alpha_o)$. Then the radius r of the N ϕ (center in y = 0 ; x = ϕ + r) and its arc-angle β for the arc from $\lambda = 0$ to $\lambda = \pi$ should suffice for the following equations:

$$(3)\ r = (x^2_0 + y_0{}^2):(2x_0); \qquad (4)\ \ \mathrm{tg}\ \beta = y:(r - x_0);$$

$$(5)\ x_0 v_0 + \frac{\rho_0{}^2}{2}\left[\alpha - \frac{1}{2}\sin 2\alpha\right] - \frac{r^2}{2}\left[\beta - \frac{1}{2}\sin 2\beta\right] = \pi \sin\phi$$

(stipulation for equality of zones). If, in (5), one first expresses r and β as functions of ϕ with x_o and y_o, using (3) and (4), and when x_o and y_o with $\rho_0 \alpha_0$ and α with the aid of (1) and (2) then equation (5) contains yet the unknown variable α, the known ϕ and the set value ρ_0 and α_0, determines, therefore, α as function of ϕ, so that now r and β are also obtained as functions of ϕ. The equations are, however so complicated, that it is not worth the calculation, especially since *true-area is not achieved*, despite having achieved area-equality of the cells and zones. An area-true circle-grid with scale-true central H is not possible.

Projections with circular prime-lines and unequally spaced central-prime-line

(SNo. 166 - 174)

The next subclass B does without the scale-true central-H. If one expects scale-true basic-lines in the first group a, then

there are three such circle-grids contributed by van der Grinten, the common characteristic of which is that the basic-line and the central-H show the same total length 2π, so that the world map border is thus a circle.

$$\rho = (\pi^2 + \lambda^2):(2\lambda)$$

thus applies for all three types, namely $\Phi = \pi \text{ tg } \frac{\mu}{2}$, where $\sin \mu = 2\,\phi:\rho$; whereas the types are differentiated by the radius rule for r. A geometric construction of r is proposed for the first type, *van der Grinten's earth map*I (SNo. 166); one may look up this construction in Zoppritz-Bludau[66]. The radius rule $r = \cot \mu$ applies for van der Grinten's earth map II (SNo. 167). This circle-grid is orthogonal. In his earth map III, van der Grinten draws the N as parallel straight lines through the dividing-points on the central-H, the projection thus becoming cylinder-circular and being classified in family II (c.f., SNo. 156).

In the following group B, with non-scale-true basic-line, one could insert a cell-equal sort (a), in which the non-scale-true central-H has a length of 2π. Concerning the border of the world map, one could expect for a number of types, e.g., the basic-line to be double so long as the central-H. This yields a = 1.1255 and for the world map border $\rho_0 = 5a : 4 = 1.407$ and $\alpha_0 = 126^\circ 52.2'$. One could again expect equal-area

of the zones for the determination of the r in one type. But here one would also (as in the case of SNo. 165) obtain a tricky equation for r as function of ϕ. The equations for this projection, No. 168, have therefore not been derived. Here also, in spite of cell-equality and zone-equality, true-area could not be achieved, since an *area-true circle-grid is not at all possible.* One can derive the proof for this in the stipulations given by Tissot-Hammer[67]. By means of elimination of ϕ and λ from the three equations:

$$\frac{\sin\psi - \psi\cos\alpha}{\sin\alpha - \cos\alpha} = \sin\phi;$$

$$x = a \operatorname{cosec}\alpha \sin\phi;$$

$$y = a(\operatorname{cosec}\alpha \cos\psi - \cot\alpha)$$

one would have to try for a circle-equation of the form $x^2 + y^2 - Ay = B$ for ϕ = constant in x and y; this is obviously impossible. The designations in Tissot-Hammer are expressed in the same manner as ours in the following forms:

$$c = a; \qquad \lambda' = \alpha; \qquad M = \rho;$$

$$\Omega = \rho\cos\alpha; \qquad M - \Omega = \Lambda;$$

$$tg = \frac{x}{y - \rho\cos\alpha}$$

The important second or (b) of this group is composed of the true-shape circle-grid of the first class, according to Lambert (1772). If, in such a shape-true circle-grid (Illustra-

tion 7, plate 1), the half central-H OP = OP' = a, PMP' the H-line λ, UDU the N-line ϕ, then the angle at which the H λ intersects the central-H $\lambda = 0$ is α; and the angle LOU = OSU is called β, where U, the intersection point of the N ϕ with the circle, is at O with the radius a and SU is the radius of the N-circle ϕ. Then the equations $\alpha = n\lambda$ and

$$tg\left(45° - \frac{\beta}{2}\right) = tg^n\left(45° - \frac{\phi}{2}\right)$$

apply for the true-shape circle-grid. For the H-circle (PMLP'), TP = ρ = a cosecα and OL = Λ = a tg $\frac{\alpha}{2}$; and for the N-circle ϕ (UMDU , SU = r = a cot β and OD = Φ = a tg $\frac{\beta}{2}$. The types are differentiated only by the fixed value n.

The reproduction rule $\alpha = n\lambda$ leads to continuously repeated reproductions of the surface of the globe, when one lets λ increase past π. Each single reproduction lies between two arcs through the pictures of the basic-points P and P', where the two arcs enclose the angle $2n\pi$. These repeated coverings of the surface of the map are congruent with each other if $n = \frac{1}{m}$ and m is a whole number. Whereas the central reproduction $-\pi<\lambda<\pi$ is double symmetrical, the side world maps are only symmetrical to the basic-line, thus belonging to our family IV, where they are mentioned under SNo. 220.

For n = 1, the grid of the H and N becomes the same as the polar grid of a transverse-axical, all circular (sterographic)

projection (SNo. 2). If one thinks of an all-circular map as a plane-perspective projection with the sighting point $(\phi_0;\lambda_0)$ originating on the globe, then the grid of H and N of the antipodal point $[\phi_0;(\lambda_0 + \pi)]$ is true-circular (family I), whereas the grid of H and N of every other global point on the great secondary circle of the point $(\phi_0;\lambda_0)$ is a true-shape circle-grid belonging to family III. Naturally, one will assign such a map to family I, as the family with greater regularity. Nevertheless, such a shape-true circle-grid with fixed value n = 1, and which as a world map fully covers the plane would be also mentioned in family III as SNo. 170 for the sake of completeness.

For $n > 1$ and with values of n between 1 and 2, the infinite straight line of the plane belongs to the picture of the meridian $\lambda = \pm 180^\circ : n$; and the whole angle-space outside of the global biangle between meridian $\lambda = -180^\circ (2 - n)$ and $\lambda = +180^\circ (2 - n)$ becomes double covered by the first single world map. Nevertheless, there have also been shape-true circle-grids with $n > 1$ proposed, namely, by Lagrange (1782). Our SNo. 169 applies for $n = \sqrt{2}$.

The somewhat more general equation

$$\operatorname{tg} \frac{\delta'}{2} = \left(\operatorname{tg} \frac{\delta}{2} \cot \frac{\delta m}{2}\right)^n$$

where $\delta = 90^{\circ} - \phi$; $\delta' = 90^{\circ} - \beta$ and δ_m is the circle of latitude reproduced as straight line, applies for the shape-true circle-grids, according to Lambert and Lagrange, rather than the equation

$$tg\left(45° - \frac{\beta}{2}\right) = tg^{n}\left(45° - \frac{\phi}{2}\right)$$

But only the shape-true circle-grids with *straight-line great secondary-cricle* belong in our family III of doubly symmetrical projections, that is

$$\delta_m = \frac{\pi}{2} \text{ and } \cot\frac{\delta_m}{2} = 1$$

In order to make these relationship more clear, Illustration 8 (plate III) shows an only simply symmetrical shape-true circle-grid after Lambert and Lagrange, in which the latitude circle $\delta_m = 60^{\circ}$ in a straight line and $n = 3/4$ is employed. This belongs to family IV as SNo. 217. The angles $\alpha = \frac{3}{4}\lambda$ lie between the meridian-circles at the polar points P and P', where there seems to be a deviation from true-shape (c.f., p. 35). It has already been mentioned (p. 35) that the shape-true conical projections (No. 39) with the equations

$$\alpha = n\lambda; \qquad \rho = tg\ \delta_m \left[tg\ \frac{\delta}{2} \cot\frac{\delta_m}{2}\right]$$

$$n = \cos\delta_m$$

with scale-true N δ_m, and their border cases, the Mercator projections No. 94 and 95, are special cases of the shape-true circle grids. As a matter of fact, Ill. 8 becomes a shape-true

conical projection, when we maintain the pole P and the numbers unchanged but allow P' to extend off into infinity, in which case all H are straight-line and all N become common-centered arcs with P as center. If we then let pole P extend off into infinity and, at the same time, let n decrease to zero, while the H become parallel equal-spaced straight lines, then we have obtained the Mercator projection.

The relationships which exist between the different forms of the shape-true circle grids are interesting, in so far as a shape-true circle-grid is always obtained again by means of *transformation with reciprocal rays* (Umwandlung mit Kehrwertstrahlen (Transformation durch reziproke Radien)). For, such a transformation assigns to each point A lying at a distance R from a fixed middle-point O a picture-point A' in the same direction from O but at a distance of $R' = c^2 : R$ (c is a fixed value), thereby picturing all angles unchanged and all circles as circles. Here, the circles passing through the point O are reproduced as straight lines, because for R = 0, $R' = \infty$, the picture of the point O thus becoming the entirety of the infinitely distant points of the picture surface. If one imagines a shape-true circle-grid map, which does not fill out the whole surface, such as our Illustration (plate III), as being reproduced by means of reciprocal rays, then three instances are possible. If the intersection point of the

reciprocal rays lies at O_1 outside of the map, then all $R > 0$, that is, all R' are finite; and the picture is just such a shape-true circle-grid map with the constant n, which also lies completely within the finite. For, all H remain finite full-circles which intersect at the picture-points of P and P' lying within the finite at the same angles as one their original. If the intersection-point of the reciprocal rays lies at O_2 within the map, however, then the picture is again a shape-true circle-grid with the constant n, the poles of which--the pictures of P and P'--lie within the finite. The new map, however, contains the infinitely distant straight line of the plane as picture of the point O_2 and leaves the inner space of the picture of the world map border unused.
If one lays, finally, the intersection-point of the reciprocal rays at the point P', then the picture of it becomes the infinitely distant straight line, whereas the picture-point of P remains within the finite. Thus, all H become straight lines and the map becomes a shape-true conical projection of the constant n. Shape-true conical projections and shape-true circle-grids with both poles within the finite are, then, reproductions of each other by means of reciprocal rays.
With respect to the transformation of all-circular (stereographic) maps by means of reciprocal rays, one finds this statement in the literature[68]: "Every true-shape projection, which reproduces all global circles as circles can be derived from

the stereographic by means of the reciprocal rays." According to that, one could assume that there were no stereographic maps which reproduce all global circles as circles. But that is not the case. The stereographic map is the only one which reproduces all global circles as circles; and it still always becomes a stereographic map when reproduced with reciprocal rays. The constant always remains 1.

The Mercator map (the special case of shape-true circle-grid with H and N as straight-lines), made by means of transformation with reciprocal rays, looks peculiar. The picture is a shape-true circle-grid, in which the world map fills out the space between two circles that lie in each other and touch each other, and the point of contiguity is the picture of both earth poles [cf., SNo.222].[69]

Our system table contains five shape-true, doubly-symmetrical circle-grids of the first class, among which there is one (SNo. 169) with $n > 1$. For that, Lagrange chose the value $n = \sqrt{2}$, because in this manner the change of the length increase in the center of the map $\phi = \lambda = 0$ is the slowest.

For $n = 2$, a doubly-symmetrical shape-true circle-grid of the first class covers the infinite plane just two times, so that it can appropriately be considered as the Riemann Double Plane with Branching Intersection. The author of this paper published[70] such a world map in 1911, although without the grid

of these circles. As a matter of fact, a map, the polar grid of which were a shape-true circle-grid of the constant $n = 2$, would have no significance. On the other hand, the polar grid consisting of confocal hyperbolae and ellipses, in which a true-shape circle-grid of the constant $n = 2$ would result for the primary and secondary lines $\phi = 0$; $\lambda = \pm 90°$, plays an important role in navigation. This likewise doubly-symmetrical polar grid is treated more closely under SNo. 189.

The smaller the constant n chosen for shape-true circle-grids, the more unequally formed the scale Φ for ϕ on the central-H. The following table contains the scales Φ on the central-H, the entire lengths of which is assumed to be = 2, for our projections No.169 to 173 with the constants $n = \sqrt{2}$, 1, 0.7049, 0.6115 and 0.5; also the line segments Λ corresponding to the values $\lambda = 45°, 90°, 135°, 180°$ on the basic-line, and the angle-contents W of the map between the border meridians of the world map at the two prime-points.

Shape-true Circle-grids (SNo. 169 - 173)

[See above noted table on page 105a following.]

Auf Kugel = on globe   Breite = latitude   Lange = longitude

Auf Karte = on map   Bogenlange = arc length

$\Lambda = \infty$ in the projection with $n = \sqrt{2}$, for $\lambda = \pm 127$ 16.8'.

On the other hand, one recognizes how unusually irregular the Φ-scale for the smallest value $n = 0.5$ is, where the interval between the $N\phi = 75°$ and $90°$ is eight times as great as the

Winkeltreue Kreisnetze (SNr. 169–178)

Auf Kugel	W	Breite $\varphi°$	15°	30°	45°	60°	75°	90°	Länge $\lambda°$	45°	90°	135°	180°
	360°	Bogen-länge φ	0,1667	0,3333	0,5000	0,6667	0,8333	1,0	Bogen-länge λ	0,5	1,0	1,5	2,0
Auf Karte			Φ	Φ	Φ	Φ	Φ	Φ		Λ	Λ	Λ	Λ
$n = \sqrt{2}$	509° 7,2'		0,1850	0,3700	0,5533	0,7311	0,9925	1,0		0,6192	2,0180	-10,1[illegible]	-1,3136
$n = 1$	360°		0,1317	0,2679	0,4142	0,5774	0,7673	1,0		0,4142	1,0000	2,4142	∞
$n = 0{,}7019$	253° 44'		0,0939	0,1912	0,3011	0,4335	0,6180	1,0		0,2850	0,6150	1,0047	2,[illegible]
$n = 0{,}6115$	220° 9,6'		0,0807	0,1664	0,2631	0,3823	0,5511	1,0		0,2440	0,5195	0,8779	1,1203
$n = 0{,}5$	130°		0,0657	0,1335	0,2168	0,3078	0,4676	1,0		0,1989	0,4142	0,6682	1,[illegible]

interval between N ϕ= 0° and 15°. Nevertheless, just this world map is probably the only drawn shape-true circle-grid in the text books (e.g., Zoppritz-Bludau, p. 180, fig. 108), because Lambert, the first author of shape-true circle-grids, proposed it with the inappropriate remark (1772) that: "It seems *most natural* that one employ n = ½." One reproduced thereby the entire surface of the globe of the earth in the form of a full circle, certainly a most *unnatural* picture, since it renders the central-meridian just as long as the equator, rather than half as long (as is the case on the globe), and it shrinks the angle-space around each pole to 180°, rather than the 360° on the globe. In this manner the polar areas are hugely stretched in the direction of the meridian.

Van der Grinten, with his projections (SNo. 156, 160 and 167) overcame this distortion only incompletely, because he also represented the globe as a full circle. For this reason, I have determined n in such a manner that the entire length of equator and central-meridian are at the proper relationship 2 : 1 to each other, in order to propose a better shape-true circle-grid projection. Then, n = 0.7049 (SNo. 171). The small table above gives the values Φ and Λ for the exact amount n = 0.7049; and for the map-grid drawn with the rounded value n = 0.7 (Illustration 9, plate IV), the amounts Φ and Λ, as well as the angle α and β, apply; their significance is the same as in

Illustrations 5 and 7 (pages 49 and 52 of the original).

Coordinates Φ and Λ and angles α and β of a shape-true circle-grid.

(n = 0.7)

[See above noted table on page 107a following.]

The grid is without doubt a better reproduction of the entire earth than Lambert's grid with n = 0.5, even than the all-circular with n = 1 and the one after Lagrange with $a = \sqrt{2}$, both of which need the infinite plane or even more yet than that for the reproduction.

One can also make the stipulation that the entire length of the central-meridian,as compared with the entire length of the equator, be decreased at the same ratio as that of the border meridian is increased. This ratio would result in 1.4303; and one would have to employ n = 0.6115. For this projection, too, (SNo. 172) the little table above, "Shape-true Circle-grids," gives a view of the process of Φ and Λ .

In the last sort of this class, the pictures of the primary-circles $\lambda = \pm \pi/2$ are two equal-spaced semi-circles, as in the projections No. 158 (globular for the hemisphere) and No. 170 (all-circular, transverse-axical). In the type of this sort, we let the mean of Φ and/or Λ of the two named projections apply for Φ and/or Λ. The thus produced combination map (SNo. 174) is Nell's modified globular projection (1852).

Koordinaten Φ und Λ und Winkel β und α eines winkeltreuen Kreisnetzes

($n = 0,7$)

$\varphi°$	$\beta°$	Φ	$\lambda°$	$\alpha°$	Λ	$\lambda°$	$\alpha°$	Λ
15	10° 23,7'	0,[illegible]	15	10,5°	0,0[illegible]10	105	73,5°	0,7[illegible]
30	21 30,4	0,1[illegible]	30	21,0	0,1[illegible]	120	84,0	0,[illegible]
45	33 13,0	0,2[illegible]	45	31,5	0,[illegible]	135	94,5	1,0[illegible]
60	46 37,0	0,4[illegible]	60	42,0	0,[illegible]	150	105,0	1,[illegible]
75	62 4[illegible],3	0,6105	75	52,5	0,[illegible]	165	115,5	1,[illegible]
90	90 0,0	1,0000	90	63,0	0,012[illegible]	180	126,0	1,9[illegible]

Polar-point maps with non-circle-shaped primary lines (SNo.175-179)

The H are not circles in the following class. Until now, only such projections of a subclass with scale-true central H have been submitted. The basic-line is also scale-true in the first group. In the first sort, the pictures of the primary-circles $\lambda = \pm \pi/2$ are again two equal-spaced semi-circles, as in projections SNo. 158, 170, 174, so that the N are determined as arcs through the three dividing-points on the H $= -\pi/2$, 0, $+\pi/2$. In the type (SNo. 175), the H are determined as the same half-ellipses as in the projection by Mollweide (No. 136). No. 175 is projection I by Fournier (1646).

The second sort includes the *all-conical* projections. The first type (No. 176) has its H so determined that all all-conical determined N should be scale-true. The equations are: $\Phi = \phi$; $r = \cot \phi$ and $\beta = \lambda \sin \phi$. This is the ordinary *polyconical projection* rendered in 1855 by the *American Coast and Geodetic Survey*. The H of the second type (No. 177) are so determined that all N intersect rectangularly. The equations of this projection read $\Phi = \phi$; $r = \cot \phi$; $\mathrm{tg}\, \beta/2 = \lambda/2 \sin \phi$. This is the *rectangular polyconical projection of the British War Office* given in 1860.

Projection No. 178, which one may call *secondary-circular* all-globular, also belongs with the group with scale-true basic-lines. In this projection, the border-meridians of the full-

globe and the N are drawn exactly as in the full-globular projection (No. 159). The other H, however, are not drawn as circles, but so that all N are equal spaced. This projection is just a little different from No. 159. Its equations are:

$$x = \frac{5\pi}{8} \sin \phi; \quad y_\pi = \pm \frac{\pi}{8} (3 + 5 \cos\phi);$$

$$\mathrm{tg} \frac{\beta_\pi}{2} = \frac{\phi - x_\pi}{y_\pi}; \qquad \beta(\lambda) = \frac{\lambda}{\pi}\beta_\pi$$

One may foresee a sort with *all-conical* projections also in the next group with *non-scale-true* basic-lines. As type for it, I have derived an area-true, all-circular projection (with scale-true central H) on the basis of the general theory of area-true, row-circular projections given in Tissot-Hammer[71] (SNo. 179), and I have drawn its grid (Illustration 10, plate V). According to Tissot-Hammer (and according to Herz), the equation

$$\frac{r}{\sin \delta}\left[\frac{dr}{d\delta}\beta - \frac{ds}{d\delta} \sin\beta\right] = \lambda + t$$

applies for all area-true, row-circular projections, where the designations of our Illustration 5 (plate I) correspond to, and t is an arbitrary function of $\delta = (90^\circ - \phi)$. In illustration 5, M is the picture-point of the global-point (λ, δ) on the N-circle with the radius r at the point S, which stands off from the basic-line at s = OS and β = OSB. If we take 1 = 0 and employ $r = \mathrm{tg}\,\delta$; $s = \frac{\pi}{2} + \mathrm{tg}\,\delta - \delta$ as must be done with the all-conical projections, then the above true-area stipulation for the determination of β yields the equations

$$\beta = \sin\beta \sin^2\delta + \lambda \cos^3\delta \text{ or } (1)\ \beta = \sin\beta \cos^2\phi + \lambda \sin^3\phi$$

and the coordinates x,y of the point M are: (2) $x = \phi + 2\cot\phi \sin^2\frac{\beta}{2}$
$y = \cot\phi \sin\beta$ (3)

$\beta = \pm\lambda$ and $dy/dx = -tg\lambda$ for $\phi = \pm \pi/2$. The projection is thus shape-true at the two polar-points $x = \phi = \pm \pi/2$. $x = 0$ and $\beta = 0$ for the points of the basic-line $\phi = 0$. The scale y on the basic-line cannot be directly calculated from the equation $y = \sin\beta / tg\,\phi = 0/0$. In a letter to me, however, F. Lange derived that for $\phi = 0$ one can put the equation for y in the form $y^3 + 6y = 6\lambda$. The derivation is as follows: in $y = \sin\beta/tg\,\phi$, with nominator and denominator differentiated according to ϕ, yields

$$y = \cos^2\phi \cos\beta \frac{\delta\beta}{\delta\phi} \quad \text{thus for } \phi = \beta = 0;\ y = \frac{\delta\beta}{\delta\phi} = \beta'$$

Also $y = \sin\beta/\sin\phi$ can be employed for $\phi = \beta = 0$ in the place of $y = \sin\beta/tg\,\phi$; thus, for $\phi = \beta = 0$, $y = \sin\beta/\sin\phi = \beta'$; β' can also be obtained according to ϕ by means of differentiation of the equation (1). This yields

$$\beta' = \frac{3\lambda \sin^3\phi \cos\phi - \sin 2\phi \sin\beta}{1 - \cos\beta \cos^2\phi} = \frac{0}{0}$$

for $\phi = \beta = 0$

By means of differentiation of nominator and denominator according to ϕ and cancellation with $\sin\phi \cos^2\phi$, one finds:

$$\beta' = \frac{6\lambda - 3\lambda\, tg^2\phi - 2\dfrac{\cos 2\phi}{\cos^2\phi}\dfrac{\sin\beta}{\sin\phi} - 2\dfrac{\cos\beta}{\cos\phi}\beta'}{\dfrac{\sin\beta}{\sin\phi}\beta' + 2\dfrac{\cos\beta}{\cos\phi}}$$

that is, for $\phi = \beta = 0$, if one employs y for $\sin\beta/\sin\phi$ and for β':

from which y as function λ can be calculated also on the basic-line $\phi = 0$.

In the table on the next page, the number values r, s, β x and y for such an area-true all-conical projection are compiled in a selection which suffices for the drawing of a grid of 15° to 15° in longitude and latitude.

The grid of this projection, corresponding to a globe diameter of 75 mm, is reproduced in illustration 10 (plate V). In this projection, it looks as though the two H $\lambda = \pm 90°$ had come together so as to form an ellipse. But that is not the case. One cannot obtain an equation, from the three determining equations (1), (2), (3) of the projection, by means of eliminating β and ϕ, which represents an ellipse equation for $\lambda = \pm \pi/2$ in x and y. Our H bends somewhat more sharply outward than the ellipse through the four axial points $\phi = 0$, $\lambda = \pm \pi/2$ and $\lambda = 0$, $\phi = \pm \pi/2$. This interesting projection combines circle-shaped parallels of latitude with true-area, and true-scale and true-shape on the central-meridian without stretching a single point out to the line. It is unfortunate, however, that in this projection the equator is about 5/4 or so long as the central meridian, rather than being two times as long. An essentially more fortunate transformation of this projection is rendered in SNo. 184.

Polar-line maps with non-circle-shaped primary-lines (SNo.180,181)

Polar line maps as in our subbranch A, in which no two N intersect each other, have up to now never been proposed. The meridianal stretching in the polar areas of those circle-grids of class I--which reproduce the earth as full circle (SNo. 166, 167, 173)--incited one to correct this deficiency by transforming the figures of the ϕ-scale in such a grid and letting some N-circle act as picture of the pole, thus changing over to a polar-line map. As an example of this, I have carried it out in SNo. 180

[The following table is found on page 112a.]

Erlauterung der Systemtabelle = Explanation of the system table

Koordinaten des flachentreuen allkegeligen Entwurfs (SNr. 179) = Coordinates of the area-true, all-conical projection (SNo. 179)

German decimal point is expressed: ","

with the *rectangular circle-grid*, which is the basis of the earth map no. II (SNo. 167) by van der Grinten. Furthermore, I have made the central-H scale-true. To do this, one only needs to rewrite that determining equation of projection 167 for the scale on the central-H, $\Phi = \pi \operatorname{tg} {}^{\mu}/_{2}$, so that it takes on the form of $\phi = \pi \operatorname{tg} {}^{\mu}/2$, and to then use this equation, in the place of the now discarded equation $\sin\mu = 2\phi : \pi$ for the calculation of μ as function of ϕ, while the equations $r = \pi \cot\mu$ and $\rho = (\pi^2 + \lambda^2) : (2\lambda)$ remain unchanged. The grid (Illustration II, plate II)

Erläuterung der Systemtabelle
Explanation of the system table

Coordinates of the area-true, all-conical projection (SNo 179)
Koordinaten des flächentreuen allkegeligen Entwurfs (SNr. 179)

(Für $\varphi = 0$ ist $\beta = 0$; für $\varphi = \frac{\pi}{2}$ ist $s = x = \frac{\pi}{2}$). Kegelradius = 1
(For ... For ...) Polar... of cone...

	$\varphi°$	0°	7,5°	15°	22,5°	30°	37,5°	45°	52,5°	60°	70°	75°	80°
	r	∞	7,5958	3,7321		1,7321		1,0000		0,5774	0,3640	0,2679	0,1763
λ°	s	∞	7,7267	3,9939		2,2557		1,7854		1,6246	1,5857	1,5769	1,5723
0°	x	0	0,1309	0,2618		0,5236		0,7854		1,0472	1,2217	1,3090	1,3963
15	β	0	1° 56,16'	3° 50,4'		7° 25,22'		10° 32,32'		12° 57,18'	14° 4,02'	14° 23,6'	
..	x	0	0,1232	0,2704		0,5292		0,8023		1,0519	1,2327	1,3[illegible]	
..	y	0,2373	0,2565	0,2499		0,2242		0,18304		0,1294	0,0885	0,0670	
30	β	0	3° 45,6'	7° 28,0'		14° 32,0'		20° 45,72'		25° 41,76'	29° 2,54'	28° 33,02'	
..	x	0	0,1472	0,2934		0,5720		0,8303		1,1043	1,2942	1,3423	
..	y	0,5207	0,4931	0,4850		0,4346		0,3545		0,2504	0,1711	0,1294	
45	β	0		10° 45,6'		21° 4,9'		30° 24,6'		33° 3,45'	41° 43,45'	43° 30,65'	44° 11,04'
..	x	0		0,3274		0,6365		0,9220		1,1709	1,3144	1,3916	1,4462
..	y	0,7157		0,6963		0,6230		0,5003		0,3559	0,2426	0,1894	0,1220
60	β	0	6° 52,8'	13° 42,0'		27° 1,5'		30° 23,7'		40° 55,9'	53° 17,92'	57° 17,33'	
..	x	0	0,1856	0,3680		0,7027		1,0124		1,2529	1,3783	1,4322	
..	y	0,9214	0,9099	0,8832		0,7870		0,6347		0,4418	0,2932	0,2233	
75	β	0	8° 10,8'	15° 19,5'		32° 24,0'		47° 42,45'		61° 10,38'	65° 28,14'	71° 13,14'	
..	x	0	0,2032	0,4121		0,7232		1,1125		1,3479	1,4521	1,4996	
..	y	1,0967	1,0908	1,0493		0,9231		0,7397		0,5063	0,3455	0,2537	
90	β	0		18° 42,0'	25° 0,9'	37° 16,5'	45° 23,1'	55° 24,0'	64° 1,82'	72° 4,8'	81° 18,0'	84° 55,8'	87° 41,6'
..	x	0		0,4589	0,6736	0,8773	1,0596	1,2176	1,3477	1,4459	1,5306	1,5508	1,5635
..	y	1,2114		1,1969	1,1340	1,0430	0,9443	0,8231	0,6899	0,5493	0,3598	0,2609	0,1762
105	β	0	10° 24,0'	20° 40,8'		41° 43,5'		62° 22,34'		82° 23,7'	93° 49,9'	95° 24,99'	
..	x	0	0,2557	0,5057		0,9630		1,3243		1,5481	1,6097	1,6162	
..	y	1,3[illegible]	1,3712	1,3271		1,1528		0,9874		0,5723	0,3632	0,2651	
120	β	0	11° 22,9'	22° 40,2'	34° 13,0'	45° 48,4'	57° 29,86'	69° 12,6'	80° 50,4'	92° 44,85'	106° 0,95'	111° 42,15'	
..	x	0	0,2[illegible]	0,5526	0,8114	1,0183	1,2439	1,4304	1,50[illegible]	1,6522	1,6863	1,6790	
..	y	1,5067	1,4[illegible]	1,4[illegible]	1,3598	1,2419	1,0999	0,9249	0,7377	0,3792	0,3199	0,2499	
135	β	0		24° 32,6'		49° 36,0'		75° 26,4'		101° 42,0'	117° 56,4'	124° 48,45'	139° 15,45'
..	x	0		0,5[illegible]		1,1730		1,5[illegible]		1,7414	1,7562	1,7229	1,6563
..	y	1,6[illegible]		1,5[illegible]		1,3191		0,9689		0,5652	0,3215	0,2200	0,1345
150	β	0	13° 3,9'	25° 12,6'	39° 32,4'	53° 7,5'	67° 3,00'	81° 21,42'	96° 1,20'	110° 49,54'	125° 37,74'	137° 45,08'	
..	x	0	0,3271	0,6[illegible]	0,9451	1,2163	1,4496	1,6409	1,7641	1,8296	1,8179	1,7721	
..	y	1,7324	1,7152	1,6[illegible]	1,5370	1,3855	1,2001	0,9846	0,7631	0,5297	0,2893	0,1502	
165	β	0	13° 54,0'	27° 45,3'		55° 25,9'		85° 59,7'		113° 36,5'	141° 7,2'	150° 34,42'	
..	x	0	0,3533	0,6912		1,2979		1,7330		1,9127	1,8690	1,8193	
..	y	1,8937	1,9247	1,7292		1,4432		0,9986		0,5020	0,2285	0,1316	
180	β	0		29° 12,0'	44° 9,6'	59° 33,0'	75° 31,62'	92° 15,9'	108° 31,3'	125° 10,2'	152° 27,6'	163° 19,6'	172° 9,42'
..	x	0		0,7353	1,0769	1,3799	1,6320	1,9249	1,9447	1,9813	1,9084	1,9336	1,7473
..	y	1,9347		1,9270	1,0919	1,4931	1,2619	0,9992	0,7217	0,4539	0,1693	0,0779	0,0241

TEXT NOT REPRODUCIBLE

German decimal point is expressed: ","

is, then, extremely easy to calculate and comfortable to draw. The following table gives some values for r and ρ at equator degrees as functions of ϕ and/or λ :

Rectangular Circle-Grid with Scale-true Central-H and Basic-Line (SNo 180)

[The above noted table may be found on page 113a following.]

Illustration 11 shows the quite nice looking rectangular circle-grid, in which the rest of the meridians and parallels of latitude appear almost spaced in equal form. The only disadvantage of this grid is that the poles are not reproduced as points, but as arcs with a length of 250.4 equator-degrees, whereas the polar lines are 360 equator degrees long on the likewise rectangular cylindrical world map, and the rectangularity is missing on the Eckert world maps with polar line length of 180 equator degrees.

The next subbranch B also offers polar lines where all N are arcs through the two points $\phi = 0$, $\lambda = \pm\pi$. These are polar-line maps, circle-grids of the second class, which are discussed in Tisso - Hammer[72]. These grids are the same as those of the first class (SNo. 158 - 174); except that now the N-lines $\phi = 0$ to $\phi = \pm 90$ are considered as H-lines from $\lambda = 0$ to $\lambda = \pm 180°$, and inversely, the H from $\lambda = 0$ to $\lambda = \pm 180°$ are now considered as N from $\phi = 0$ to $\phi = \pm 90°$. Such maps can only be useful for representing smaller parts of the globe. They cannot be used

Rechtschnittiges Kreisnetz mit maßtreuer Mittel-H und Grundlinie (SNr. 180)

$\Phi = \varphi$	r	$\Lambda = \lambda$	ϱ	$\Lambda = \lambda$	ϱ
15°	1072,70°	15°	1062,60°	105°	203,79°
30	524,92	30	555,00	120	195,00
45	337,46	45	382,50	135	187,50
60	240,00	60	300,00	150	163,00
75	178,50	75	253,50	165	160,72
90	135,00	90	225,00	180	160,00

as world maps, since the poles are stretched out into lines of far more than equator length, and, furthermore, they let the border-meridians $\lambda = \pm 180°$ shrink down to points, when the circle-grid of the first class is a point map, or, when it is a polar-line map, to a small arc. An example of the latter type is given in Tissot-Hammer (fig. 17), a shape-true circle-grid of the second class, for which the following equations apply:

$$\rho = \cot\alpha; \quad \Lambda = \operatorname{cosec}\alpha - \cot\alpha, \text{ where log nat } \cot\frac{\alpha}{2} = \lambda$$

$$r = \operatorname{cosec}\beta; \quad \Phi = \operatorname{cosec}\beta - \cot\beta, \text{ where } \beta = \text{log nat } \cot\frac{\delta}{2};$$

$$\delta = \frac{\pi}{2} - \phi$$

In this grid, SNo. 181, the parallels of latitude are arcs through the points $x = 0$; $y = \pm 1$. The map, expanded to the latitudes $= \pm 85$ 3.08', fills up the whole plane, with the exception of two small circles with radius 0.0866 at the points $x = 0$; $y = \pm 1.0037$. The small circles are the pictures of the meridians $\lambda = \pm 180$, whereas the meridian $\lambda = 0$ is the infinitely long straight line $y = 0$. The equator has an entire length of 1.8266. The rest of the circles of latitude keep getting greater with the increasing latitude, until the circle of latitude ± 85 3.08' consists of the infinitely distant straight line and the straight line $x = 0$, with exception of the piece $81.0037 < y < + 1.0037$. In the grid by Tissot-Hammer drawn only up to $\pm 75°$

latitude, also, 75° on the central-meridian require more than double the room required by 90° on the equator.

FAMILY III, BRANCH B: CURVED-SYMMETRICAL, NOT ROW-CIRCULAR (SNo. 182 - 196)

Subbranch A: Transformed, Order a: Expanded (SNo. 182 - 184)

In Branch B, the N are not a system of circles, but are curves among which in single instances a circle may appear. The subbranch A contains projections, which are produced by means of *transformation* of already discussed doubly-symmetrical grids. It was already mentioned on p. 13 that the grid of the map, central-point in the transverse-axical central-perspective projection (SNo. 1), corresponds to the characteristics of family I (true-circular), but that the polar-grid belongs with family III (curved-symmetrical). The same applies for every other radial transverse-axical projection. They may all, according to their polar grid, be considered as members of family III. They and all projections of families II and III may be thought of as original forms which can be transformed.

The *first order* of these transformed projections is an *affine reproduction*, in which the ratio of the coordinates x : y changes uniformly for each map-point, the map thus being expanded or even compressed to various degrees in the x- and y-directions. The geometric order of all curves in such a transformation remains unchanged. All straight lines of the original map are also straight lines in the expanded map; all circles remain curves of the second order (cone-section) and in general become ellipses. Thus, if the polar grid of

transeverse-axical radial projections is expanded in the *first class* , then the great circles of the map center remain straight lines and their secondary-circles become conic sections. If there is an expansion of the original map in one direction in the same ratio as a compression in the other direction, then the area sizes remain unchanged, so that a new area-true projection is produced from an area-true original projection. However, true-shape is not maintained in the expansion, but the radial transverse-axical projections which are shape-true in the center do get two shape-true points rather than just this one.

It is for just this reason that the expanded transformation (SNo. 182) of the transverse-axical center-perspective projection (SNo. 1) is of significance. Because it has gained straight lines, the new projection is also *straight-directional* ; and since it now has two shape-true points, it is valuable, e.g., as radio directions finding map with two preferred radar stations. If the equations $x = tg\,\phi \sec\lambda$; $y = tg\,\lambda$ apply for the transverse-axical projection SNo. 1, then $x_1 = m\,tg\,\phi \sec\lambda$; $y_1 = n\,tg\,\lambda$ applies for the expanded transformation. The meridians remain parallel straight lines, the circles of latitude remain hyperbolae. True-shape is produced,

if m>n, at the central-meridian at the two points $x_1 = 0$; $y_1 = \pm\sin L$, where $\cos L = n : M$

if m<n, " " equator " " " " $y_1 = 0$; $x_1 = \pm\sin L$, where $\cos L = m : n$

Naturally, true-shape can be obtained in such a projection also for two arbitrary map-points A and B; one has only to use an oblique-axical center-perspective projection as original, where the dividing point of arc A B is the map center, while the great circle A B takes the place of the equator. Herle drew the first straight-directional non-center-perspective projection [the designation "gnomonic" is not appropriate for straight-directional projections with two shape-true points] in the 1890's; merely in order to save space, he compressed the meridians of a transverse-axical gnomonic map at the same ratio. He did not realize that he had obtained two shape-true points in this manner. Littlehales [73] came up with a second such map in 1899. That other straight-directional maps with two shape-true points and more advantageous distortion are produced by means of collinear reproduction of gnomonic maps was shown by Maurer [74] in 1914; at this time he published a map of this nature. Further literature: Wedemeyer [75], Immler [76], Thorade [77] and Maurer [78].

As example of an area-true projection of this class, we may consider the not yet proposed expansion of the polar-grid of a transverse-axical area-true radial projection by Lambert (SNo. 23), because this new projection (SNo. 183) exhibits true-area with two shape-true points, and, under certain circumstances, far more advantageous distortion ratios than the--quite justifiedly--much used Lambert Projection.
The equations $x = 2 \sin \delta/2 \cos \alpha$; $y = 2 \sin \delta/2 \sin \alpha$ apply

for its polar grid, where δ is the arc-interval of the global-point from the prime-point, thus $\cos\delta = \cos\phi\cos\lambda$, and is the azimuth of the global point from the prime-point, thus $tg\,\alpha = \sin\lambda\cot\phi$. In the expanded map, which should remain area-true, $x_1 = m\,x$ and $y_1 = y : m$. For the greatest shape-distortion 2 w,

at each equator-point, $\sin w = \left(m^2 - \cos^2\frac{\lambda}{2}\right) : \left(m^2 + \cos^2\frac{\lambda}{2}\right)$

at each central-meridian-point, $\sin w =$

$$\left(1 - m^2\cos^2\frac{\phi}{2}\right) : \left(1 + m^2\cos^2\frac{\phi}{2}\right)$$

For SNo. 23, m = 1. That more favorable distortion occurs for $m > 1$, has been shown on a map of Africa, according to both projections. Such a Lambert Projection can be found, for example, in Andree's Hand Atlas, 8th edition, 1928. The map extends at the central-meridian (20° east) from about 35° north to 35° south; and, at the equator, which passes through Africa from 9° east to 43° east, the uppermost shape-distortion 2 w to 28° difference in latitude is checked from the central-meridian. The following table for 2 w shows that an expanded map (m = 1.0086) does an overall better job for the area than the Lambert Projection with m = 1.

Uppermost Angle Distortion 2 w for Two Area-true Maps of Africa; Lambert ($m = 1$), expanded ($m = 1.0086$)

[The above noted table may be found on page 120a.]

On the expanded map the values of 2 w at the central-meridian-ends and the equator-ends are equal (4.4°) and less than at the meridians on the Lambert Map (5.4°). The central-meridian on the expanded map has two shape-true points ($\phi = \pm 15°$), between which the angle error remains under 1°.

When *row-circular projections are expanded*, the N-circles become ellipses. I have derived such a projection in SNo. 184, an area-true transformation of the all-conical, area-true projection No. 179. When one so freely gives up the characteristics of the all-conical, one obtains, then, two shape-true points $\lambda = 0$ on the equator, rather than one, and can strongly reduce the distortion of the great forms by letting the entire length of the equator become double so long as the central-meridian. *Illustration 12* (plate IV) shows this rather pleasing grid of an area-true world map with polar points, equal-spaced central meridian, elliptical parallels of latitude and two shape-true points at $\lambda = \pm 46.3°$ on the equator, which is double so long as the central-meridian. The equations for this projection are the same as those for SNo. 179, except for the factor n for y and 1:n for x, whereby n^2 must be twice the ratio of the lengths of the central-meridian : equator of the projection No. 179,

Höchstwinkelverzerrungen $2w$ für zwei flächentreue Karten von Afrika

Lambert ($m = 1$), gedehnt ($m = 1,0086$)

φ	λ	$m = 1$	$m = 1,0086$
0°	8° W u. 48° O	5° 27,4'	4° 23,4'
0	5° O u. 35° O	0 50,0	1 58,1
0	20° O	0 0,0	0 59,1
15° N u. 15° S	20° O	0 59,0	0 0,0
35° N u. 35° S	20° O	5 25,5	4 26,6

that is, $n^2 = 2 \cdot 1.5708 : 1.9347$ or $n = 1.2743$. The equations for No. 184 are thus:

(1) $\beta = \sin\beta\cos^2\phi + \lambda\sin^3\phi$; (2) $x = \frac{1}{n}\left(\phi + 2\cot\phi\sin^2\frac{\beta}{2}\right)$

(3) $y = n\cot\phi\sin\beta$; $n = 1.2743$.

The following table gives the coordinates according to the formulae for x and y. β has the same values as in the table for projection SNo. 179. $x = \frac{\pi}{2} : 1.2743 = 1.2327$ for $\phi = \frac{\pi}{2}$, that is, half as great as y for $\phi = 0°$; $\lambda = 180°$.

Coordinates of the Area-true Projection No. 184 (expanded, from No. 179): [See page 121a following.]

For finding the shape-true points on the equator, we use the way of thinking of F. Lange for deriving the scale-distribution on the equator of the original projection No. 179. In place of the equation there, $y^3 + 6y = 6\lambda$, we have for No. 184 the equation $\left(\frac{y}{n}\right)^3 + 6\frac{y}{n} = \lambda$

True-shape exists on the equator there, where, for $x = y = \beta = 0$,
$\frac{\delta\lambda}{\delta y}$ $(\phi = 0) = \frac{\delta x}{\delta\phi}$ $(\lambda = \text{constant})$

For $\phi = \beta = 0$, now follows from equation (3): $\frac{y}{n} = \frac{\sin\beta}{\sin\phi} = \frac{0}{0}$,
and for $\phi = \beta = 0$ by means of differentiating denominator and nominator according to ϕ: $\frac{y}{n} = \frac{\delta\beta}{\delta\phi}$.

Differentiating from equation (3) yields

$$\frac{\delta y}{\delta\lambda} = n\cot\phi\cos\beta\frac{\delta\beta}{\delta\lambda}$$

Koordinaten des flächentreuen Entwurfs Nr. 184 (gedehnt aus Nr. 170)

λ°		φ = 0°	7,5°	15°	22,5°	30°	37,5°	45°	52,5°	60°	67,5°	75°	90°
0	x	0	0,1097	0,2054		0,4109		0,6164		0,8218	0,9257	1,0263	1,0358
15	x	0	0,1095	0,2122		0,4223		0,6165		[illegible]	[illegible]	1,0340	
„	y	0,3279	0,3270	0,3184		0,2557		0,2933		[illegible]	[illegible]	[illegible]	
30	x	0	0,1192	0,2302		0,4544		0,6673		0,8670	0,9621	1,0521	
„	y	0,6636	0,6217	0,6180		0,5538		0,4518		0,3205	0,2151	0,1650	
45	x	0		0,2560		0,5019		0,7211		0,9181	1,0315	1,0313	1,1319
„	y	0,9158		0,8878		0,7949		0,6159		[illegible]	0,3051	0,2336	0,1596
60	x	0	0,1156	0,2888		0,5515		0,7541		0,9822	1,0[illegible]17	1,1239	
„	y	1,1742	1,1595	1,1263		1,0023		0,9270		0,5030	0,3813	0,2873	
75	x	0	0,1631	0,3239		0,6228		0,5739		1,0572	1,1[illegible]6	1,1598	
„	y	1,3815	1,3772	1,3371		1,1635		0,9128		0,6122	0,1115	[illegible]	
90	x	0		0,3691	0,5303	[illegible]	0,8316	0,9555	1,0576	[illegible]	1,2017	1,2169	1,2256
„	y	1,5919		1,5349	1,4116	1,3367	1,2033	1,0159	0,5791	[illegible]	[illegible]	0,3321	0,2255
105	x	0	0,2008	0,3989		0,7557		1,0305		1,2110	1,2653	1,2683	
„	y	1,7604	1,7472	1,6912		1,4680		1,1303		0,7202	0,1626	0,3378	
120	x	0	0,2269	0,4336	0,6267	0,8226	0,9732	1,1226	1,2259	[illegible]	1,3236	1,3153	
„	y	1,9225	1,9008	1,8497	1,7315	1,5825	1,4093	1,1914	0,9311	0,7522	0,1157	0,3211	
135	x	0		0,4695		0,6892		1,2012		[illegible]	1,3782	1,3575	1,3255
„	y	2,0713		1,9767		1,6903		1,2335		0,7152	0,1037	0,2804	0,1715
150	x	0	0,2567	0,5037	0,7417	0,9546	1,1375	1,2577	1,3511	[illegible]	1,1205	1,3906	
„	y	2,2050	2,1936	2,0901	1,9585	1,7656	1,5293	1,2531	0,9721	0,6878	0,3372	0,2256	
165	x	0	0,2775	0,5424		1,0135		1,3599		1,5010	1,4673	1,4206	
„	y	2,3307	2,3252	2,2126		1,9391		1,2726		[illegible]	0,2511	0,1678	
180	x	0		0,5692	0,8155	1,0823	1,2308	1,1321	1,5261	[illegible]	1,4576	1,4359	1,5712
„	y	2,4651		2,3201	2,1483	1,9026	1,6050	1,2733	0,9156	[illegible]	0,2141	0,0931	0,0307

to which is added from equation (1) $\frac{\delta\beta}{\delta\lambda} = \frac{\sin^3\phi}{1 - \cos\beta\cos^2\phi}$

so that from both equations the result is:

$$\frac{\delta y}{\delta\lambda} = \frac{n^2 \sin^2\phi}{1 - \cos\beta\cos^2\phi} = \frac{0}{0} \qquad \text{for} \qquad \phi = \beta = 0$$

By means of differentiating of nominator and demonimator according to ϕ and cancellation by $\sin\phi\cos\phi$, one obtains

$$\frac{\delta y}{\delta\lambda}(\phi = \lambda) = \frac{2n}{2 + \frac{\sin\beta}{\sin\phi}\frac{\delta\beta}{\delta\phi}} = \frac{2n^3}{n^2 + y^2}, \qquad \frac{\sin\beta}{\sin\phi} = \frac{\delta\beta}{\delta\phi} = \frac{y}{n} \quad \text{for } \phi=\beta=0$$

In order to obtain $\frac{\delta x}{\delta\phi}$, we differentiate equation (2), which we write in the form $x = \frac{\phi}{n} + \frac{y}{n}\,\mathrm{tg}\,\frac{\beta}{2}$ according to ϕ and obtain

$$\frac{\delta x}{\delta\phi} = \frac{1}{n} + \frac{1}{n^2}\frac{\delta y}{\delta\phi}\,\mathrm{tg}\,\frac{\beta}{2} + \frac{y}{2n^2\cos^2\beta}\frac{\delta\beta}{\delta\phi},$$ which results for

$\phi = \beta = 0$ in: $\frac{\delta x}{\delta\phi} = \frac{1}{n} = \frac{y^2}{2n^3} = \frac{2n^2 + y^2}{2n^3}$

True-shape exists for $\frac{\delta x}{\delta\phi} = \frac{\delta y}{\delta\lambda}$, that is: $\frac{2n^2 + y^2}{2n^3} = \frac{2n^3}{n^2 + y^2}$

from which results $\frac{y}{n} = \sqrt{2(n-1)}$. If we employ this value of in the equations which applies for the equator,

$\left(\frac{y}{n}\right)^3 - 6\frac{y}{n} = 6\lambda$, then one finds for λ_w of the shape-true points

$\lambda_w = \frac{n+2}{3}\sqrt{2(n-1)}$, which with $n = 1.2743$ yeilds

$\lambda_w = \pm 0.8084 = \pm 46°19'$.

If one compares the reproductions 10 and 12 with the two area-true projections 179 and 184, then one sees that rectangular intersection of the N and H in No. 179 beyond the basic-line occurs only on the central-H, but in No. 184 beyond the central-H at least at higher latitudes, for the two values

$\pm\lambda_r$, too, which are different from 0. By elimination of from equations (2) and (3), one finds for the N-line the ellipse-equation:

$$\left(\frac{y}{n\ \cot}\right)^2 + \left(\frac{x - \frac{\phi + tg\ \phi}{n}}{\frac{\cot\ \phi}{n}}\right)^2 = 1$$

from this, one can, by comparison with the usual ellipse-equation $\left(\frac{y}{b}\right) + \left(\frac{x - c}{a}\right) = 1$ read off the semi-axes a and b as well as the inclination of the ellipse-center y = 0; x = c. By employing the values a b c in the well-known equations for the tangent-direction at the ellipse-point (x,y),

$$\frac{dy}{dx} = -\frac{b}{a}\ \frac{(x - c)}{\sqrt{a^2 - (x - c)^2}}$$

also results. This $\frac{dy}{dx}$ must be, at places of rectangular intersection of H and N, the reciprocal value of that $\frac{dy}{dx}$ or $\left(\frac{dy}{d\phi} : \frac{dx}{d\phi}\right)$, which one finds for the meridian-equation λ ; that is, by differentiating the equations (2) and (3) according to ϕ with constant λ, while β applies as function of ϕ, according to which one must eliminate β with the aid of equation (1). These extremely complicated formulae have, however, never been carried out.

SUBBRANCH A: TRANSFORMED; ORDER b: AITOFF-ized (AITOFFIERT) (SNo. 185, 186)

This order contains transformed, double-symmetrical projections, their process of transformation going back to Aitoff. Aitoff began with the polar grid of the hemisphere in the transverse-axical radial projection (Mercator, SNo. 24), left the x coordinate for each point (ϕ / λ) unchanged, but doubled the y coordinate and gave the thus obtained point the designation (ϕ /λ), so that a reproduction of the full global grid was produced. In somewhat generalized form, this process, which one might call *Aitoff-izing* might be defined as follows: From the hemisphere-grid of a doubly-symmetrical original projection I with cooridinates $x(\phi/\lambda)$ and $y(\phi/\lambda)$, one obtains, by means of Aitoff-izing, a doubly-symmetrical projection II with the coordinates $X(\phi/2\lambda) = mx(\phi/\lambda)$ and $Y(\phi/2\lambda) = 2ny(\phi/\lambda)$. Thus, Aitoff-izing is an expanding (as in the previous order a), but with the following refiguring of the meridians. If the original projection had a shape-true point $\phi = \lambda = 0$, then this point remains shape-true even after the Aitoff-izing, if the ratio of increase $\frac{2n}{m}$ on the equator and central-meridian is equal to the refiguring number 2, that is, $m = n$. Otherwise two shape-true points as with the expanding are produced, rather than the one, in which case the equator is increased n times and the central-meridian m times.

SNo. 185 renders *"Aitoff's Planisphere"*. Aitoff produced it from the transverse-axical projection (SNo. 24), which

employed m = n = 1. The equations

$$\cos \sqrt{x^2 + y^2} = \cos \lambda \cos \phi; \quad y = x \sin \lambda \cot \phi$$

apply for the starting projection; from these equations result the equations for No. 185:

$$X(\phi/2\lambda) = x(\phi/\lambda); \quad Y(\phi/2\lambda) = 2y(\phi/\lambda)$$

When this transformation takes place, the straight-line characteristic of the great circles through the point $\phi = \lambda = 0$, as well as the general central-equal distance, are lost. But true-scale on the equator and central-meridian and true-shape at point $\phi = \lambda = 0$ are maintained. The world map border is an ellipse with an axis length π and 2π .

Hammer pointed out [79] that area-similarity is maintained when Aitoff-izing an area-true map. As far as the uniform-globe is concerned, true-area is kept, if 2mn = 1 in the transformation equations. Hammer himself obtains the Hammer Planisphere (SNo. 186) by Aitoff-izing the area-true radial transverse-axical projection by Lambert (SNo. 23) with the values $m = n = \frac{1}{\sqrt{2}}$. The equations that apply for the original projections are $x^2 + y^2 = 4 \sin^2 \frac{\delta}{2}$; $y = x \sin \lambda \cot \phi$ where $\cos \delta = \cos \phi \cos \lambda$, thus $x^2 + y^2 = 2(1 - \cos \phi \cos \lambda)$ From this, the result for No. 186: $X(\phi/2\lambda) = x(\phi/\lambda):\sqrt{2}$ and $Y(\phi/2\lambda) = \sqrt{2}\, y(\phi/\lambda)$.

True-area in the whole map and true-shape at point $\phi = \lambda = 0$ are maintained. The world map border becomes an ellipse with the semi-axes $\sqrt{2}$ and $2\sqrt{2}$. In the paper of the American <u>Coast and Geodetic Survey</u> [80], the Hammer Planisphere is incorrectly attributed to Aitoff.

There is little use in Aitoff-izing other doubly-symmetrical projections, if one does not want to attain more favorable shape relationships on an area-true projection, for example, by means of $m \gtrless n$ and $2mn = 1$, similarly to how SNo. 183 was produced from SNo. 23. A cylinder-circular projection, Aitoff-ized, becomes itself, if y is in equal ratio to λ for ϕ = constant, the projection thus being secondary-circle-spaced. This applies in family II for the branch A. Thus, only the non-secondary-circle-spaced projections, such as SNo. 153-157 by Apianus, Glareanus and van der Grinten, would yield new forms through Aitoff-ization. One could Aitoff-ize the globular projection by Nicolosi (SNo. 158), for example, and it would be easy to draw. Its circle-shaped H would become ellipses; its central-H and basic-line would remain scale-true. But the world map would be less advantageous than the full globular projection No. 159 because of the shrinking together of the angle-space at the pole to 180°. It would be the same matter, if one were to Aitoff-ize the all-conical area-true projection No. 179, the expanded transformation (No. 184) of which appears essentially more appropriate.

SUBBRANCH A: TRANSFORMED; ORDER c: WITH OTHER MERIDIAN SPACING (SNo. 187, 188)

In the last order of the subbranch, the meridian curves remain the same as in the original projection; only their distribution is changed. It is simplest to make the meridians

equal-spaced. Two examples of this class should suffice. SNo. 187, the *all-globular* projection, has the circle-shaped H in common with the full-globular No. 159, but spaces them equal-shaped. This projection was proposed by the author of this paper[65] in 1922 and has been found useful for world maps[81]. It has scale-true equator and central-meridian and circle-shaped equal-spaced meridians. The arc of the meridian λ has the radius $\rho = (\pi^2 + 4\lambda^2) : (8\lambda)$, the center-point $x = 0$; $y = \rho - \lambda$, and the regularly spaceable angle-opening y from pole to equator, where $\sin z = \pi : (2\rho)$. The coordinates of the picture-point for the global point (ϕ λ) are thus

$$x = \rho \sin \frac{2\phi z}{\pi} ; \quad y = \lambda + 2\rho \left(\sin \frac{\phi z}{\pi}\right)^2$$

The following table gives the values of ρ and z for intervals of 15° from λ. Table: Coordinates ρ and z of the All-globular Projection (SNo. 187): [See page 127a following.]

The projection, SNo. 188, by Schmidt (1803) shows the same meridian curves as the area-true projection by Molleweide (No. 135), both of which correspond to the ellipse equations

$$\frac{x^2}{2} + \frac{\pi^2 y^2}{8\lambda^2} = 1$$

where there is true-area with the uniform globe. In Schmidt's projection, the ellipse arcs may be spaced regularly, proportional to lattitude ϕ. If one introduces a running variable ϕ and employs $u^2 = 2\cos^2\psi + \frac{8\lambda^2}{\pi^2}\sin^2\psi$

the arc length Q of the quarter ellipse from equator to pole for the meridian λ is

$$Q = \int_0^{\pi 2} u \, d\psi$$

Koordinaten ϱ und z des allglobularen Entwurfs (SNr. 167)

λ	15°	30°	45°	60°	75°	90°
ϱ	4,4171	2,6150	1,9685	1,7017	1,5970	[illegible]
z	15° 55,5'	[illegible]	55° 7,8'	67° [illegible]	79° 36,7'	90° 0,0'
λ	105°	120°	135°	150°	165°	180°
ϱ	1,5995	[illegible]	1,7017	[illegible]	1,8483	1,9235
z	95° 47,8'	105° 13,6'	112° 37,2'	115° 17,5'	124° 48,6'	125° 52,4'

The number value for Q can be found from the Legendre Tables of elliptical integrals. The arc length $\sigma_1 = 2\phi_1 Q:\pi$ corresponds, then, with the latitude ϕ_1, on the Schmidt map at the same meridian from the equator to the picture point (λ ϕ_1). One determines according to the integral tables, through which upper integral limit ϕ_1, the value σ_1 $\int_0^{\psi_1} \psi$ is to be attained. Then the coordinates of the picture point being sought are $x = \sqrt{2} \sin \phi_1$; $y = \frac{2\lambda\sqrt{2}}{\pi} \cos \phi_1$

The true-area of the Mollweide Projection is lost in this transformation; only cell-equality is maintained and all meridians are equal-spaced. True-shape exists at the point $\phi = \lambda = 0$. In Tissot-Hammer's[82] description of this projection, the central-meridian and equator are accepted as scale-true, thus rejecting area-equality with the total surface of the uniform globe. Then only cell-equality exists. All border-meridians of the map are considered there $\lambda = \pm 90°$.

SUBBRANCH B: SHAPETRUE; ORDER a: WITH ALGEBRAIC GRID LINES

(SNo. 189-192)

A second subbranch of the double-symmetrical, non-row circular projections consists of the shape-true projections. True-shape is generally determined by means of differential equations, as they have been derived in Tissot-Hammer.[83] In such projections, the N are hosts of curves according to the equation $F(x,y,p) = 0$, where p is a parameter dependent on ϕ; and the meridian H are the accompanying host of equal-rectangles (orthogonal trajectories) according to the equation $F_1(s,y,q) = 0$, where q is now a parameter dependent on λ. The functions F and F_1 are not supposed to be circle-equations in this subbranch, but fulfill the stipulations for being doubly symmetrical; that is, if p_+ and p_- correspond with opposite equal ϕ-values and q_+ and q_- correspond with opposite equal λ-values, then $F(x,y,p_+) = F(-x,y,p_-)$ and $F(x,y,q_+) = F(x,-y,q_-)$ must follow. For shape-true projections it is often appropriate to use the Mercator function

$$\varepsilon = \log \text{nat} \operatorname{tg} \left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$

with its differential

$$d\varepsilon = \sec \phi \, d\phi$$

for the designation of the global point (ϕ/λ) rather than ϕ, in other words, the x-coordinates of the corresponding point of the Mercator map.

From the equations for F and F one obtains x and y as functions of p and q and can form the functions:

$$U = \sqrt{\left(\frac{\delta x}{\delta p}\right)^2 + \left(\frac{\delta y}{\delta p}\right)^2}\;; \qquad V = \sqrt{\left(\frac{\delta x}{\delta q}\right)^2 + \left(\frac{\delta y}{\delta q}\right)^2}$$

(which must be employed in Tissot-Hammer, p. 202, line 10, in place of the unsatisfactory equations

$$U = \sqrt{\left(\frac{\delta x}{\delta \xi}\right)^2 + \left(\frac{\delta y}{\delta \xi}\right)^2}\;; \qquad V = \sqrt{\left(\frac{\delta x}{\delta \lambda}\right)^2 + \left(\frac{\delta y}{\delta \lambda}\right)^2} \;!)$$

The stipulation for true-shape, which must accompany the already fulfilled stipulation for rectangularity in the equations for F and F , reads then:

$$U\frac{dp}{d\xi} = V\frac{dq}{d\lambda}$$

After dividing away a somewhat equal available factor z depending on p *and* q on both sides, this equation can be stated as

$$P\,\frac{dp}{d\xi} = Q\,\frac{dq}{d\lambda}$$

where P depends only on p and Q depends only on q. Now the sought-after dependence between p and ξ on the one hand and between q and λ on the other hand results by means of integrating the two differential equations

$$P\,\frac{dp}{d\xi} = c \quad \text{and} \quad Q\,\frac{dq}{d\lambda} = c$$

where c must be employed both times as equal to the same fixed value.

In the two projections SNo. 189 and 190, the N and H are curves of the second order. In No. 189 the N are focal-point-equal (confocal) ellipses according to the equation

$$F: \frac{y^2}{p^2} + \frac{x^2}{p^2 - 1} = 1$$

and the H as rectangularly-equal to the N are the focal-point-equal hyperbolae according to the equation

$$F_1: \frac{y^2}{q^2} - \frac{x^2}{1 - q^2} = 1$$

the focal points of which are: $x = 0$; $y = p = q = 1$. For x and y as functions of the parameters p and q the equations F and F yield $x = \sqrt{p^2 - 1}\ \sqrt{q^2 - 1}$; $y = pq$

One thus finds:

$$U = \sqrt{\frac{p^2 - q^2}{p^2 - 1}}; \qquad V = \sqrt{\frac{p^2 - q^2}{1 - q^2}} \qquad \text{and} \qquad Z = \sqrt{p^2 - q^2}$$

The differential equations

$$\frac{1}{\sqrt{p^2 - 1}} \frac{dp}{d\xi} = c \text{ and } \qquad \frac{1}{\sqrt{1 - q^2}} \frac{dq}{d\lambda} = c$$

are to be integrated for the determination of the dependence of the parameter p on ξ and of the parameter q on λ . One obtains the projection SNo. 189 with the fixed value c = 1. $q = \sin\lambda$ results in

$$\int \frac{dq}{\sqrt{1 - q^2}} = \int d\lambda$$

When d ϕ is employed for $d\xi = \sec\phi\ d\phi$

the other equation reads:

$$\frac{dp}{\overline{p \quad - 1}} = \frac{d}{\cos}$$

which is identically fulfilled for $p = \sec \phi; \sqrt{p^2 - 1} = tg\ \phi$

and $dp = \frac{\sin \phi}{\cos^2 \phi} d\phi$

Thus, the determining equation for x and y results for the projection: $x = \sqrt{p^2 - 1} \sqrt{q^2 - 1} = tg\ \phi \cos \lambda$ $y = pq = \sec \phi \sin \lambda$

Maurer[84] has presented another derivation of this projection according to its important characteristic which gives it great importance for navigation. All straight lines of the map surface are pictures of azimuth-equals of the globe, that is, each is a curve that has a determined fixed direction-point lying out from all its points on the same azimuth. Since there is a complex of ∞^3 azimuth-equals on the globe, a multiple by three, the bundle of all the only ∞^2 straight lines of the plane can only reproduce a multiple by two, ∞^2 azimuth-equals; and these are the ∞^2 azimuth-equals whose direction-points lie on the central-meridian $\lambda = y = 0$ reproduced as a straight line. Since the map is shape-true, the invariable azimuth of each azimuth-equal can also be read off as its angle of intersection with central-meridians, for which reason the seamen use this map as the so-called Weir Azimuth Diagram. Naturally, one must read the invariable azimuth angle at the central-meridian on the same side as with all other meridians, and not on the opposite side, as

A. Wedemeyer[85] did in an incorrect understanding of the term "counter-azimuthal" coined by Hammer. Hammer[41] had used the term *counter-azimuthal* in referring to projections which reproduce the azimuth-equals of only one direction-point as straight lines intersecting themselves at its picture-point at the proper angles; Hammer did this after Craig had referred to such a map which was not shape-true in the rest of the map space and had Mecca as direction-point (cf. SNo. 200). According to this definition, projection No. 189 is also counter-azimuthal, since it fulfills the stipulation for infinitely many direction-points, and since the fulfilling of this stipulation at only one direction-point already makes it counter-azimuthal. It is the *shape-true azimuth-equal* map, in which all straight lines are azimuth equals. Its equations are found mentioned in Littrow[86], but he asserted incorrectly that its meridians and circles of latitude were hyperbolae. Maurer published the first more exact examination of this map in 1905[84] and 1911[70] and included such maps. The world map (Illustration 13, plate VIa and b) fills out the surface two times as the Riemann Double Surface with the equator and its length increases, the meridians $\lambda = \pm 90°$ as branching-intersections. The picture of both earth poles is the infinitely distant straight line of the plane. Whereas the polar grid consists of hosts of confocal ellipses and hyperbolae, the primary- and secondary-circles of the pair of points $\phi = 0$ and $\lambda = \pm 90°$ produce and Lambert-type circle-grid with the constant n = 2, the context of which has

already been presented by Wedemeyer[87] in 1918. It would not be appropriate, however, to place the azimuth-equal map into the system as a Lambert circle-grid, since the concept of the azimuth is linked to the meridians and the straight lines of the map surface are azimuth-equals only when the meridians are hyperbolae. A polar grid which is a Lambert circle-grid is not an azimuth-equal map. For this reason the designation *Littrow-Maurer Azimuth-equal Map* for projection SNo. 189 seems justified.

If, in the grid No. 189, one considers the ellipses as meridians and the hyperbolae as circles of latitude, then one has the projection (SNo. 190) by Firoini[83]. Its equations, then, are: $x = \sec \lambda \sin \phi$; $y = \mathrm{tg}\, \lambda \cos \phi$
The map consists of two full planes, the center points of which are the points ($\phi = 0$; $\lambda = 0$) and ($\phi = 0$; $\lambda = 180°$)
The equator is an infinitely long double straight line; and the infinitely distant straight line is the picture of the meridian $\lambda = \pm 90°$. The poles are reproduced as straight lines, $y = 0$; $x > 1$ and $y = 0$; $x < -1$, while the central-meridian is pressed together on the line segment $y = 0$; $-1 < x < 1$. This shape-true projection is not an azimuth-equal map.

The H and N of the projections SNo. 191 and 192 are curves of a higher order. No. 191 is a transformation of No. 189 and is presented because the poles in No. 189 fall infinitely distant and the shape-true azimuth-equal map is thus not usable in the polar areas. Its transformation by means of reciprocal

rays brings the pole into the center of the map and maintains true-shape, while the straight azimuth-equals of No. 189 become likewise well usable circles.

The equations for the coordinates x,y of the projections SNo. 191 are formed from those for No. 189, $x = \mathrm{tg}\,\phi \cos\lambda$; $y = \sec\phi \sin\lambda$ through the transformation equations

$$\frac{y_1}{x_1} = \frac{y}{x} = \frac{\mathrm{tg}\,\lambda}{\sin\phi}\ ; \quad y_1^2 + x_1^2 = \frac{1}{x^2 + y^2} = \frac{1}{\mathrm{tg}^2\phi + \sin^2\lambda}$$

and read $$x_1 = \frac{\cos\lambda \sin\phi \cos\phi}{1 - \cos^2\lambda \cos^2\phi}\ ; \quad y_1 = \frac{\sin\lambda \cos\phi}{1 - \cos^2\lambda\cos^2\phi}$$

The equation for the $H(\lambda)$ and/or $N(\phi)$ is obtained when one eliminates from the equation

$$\frac{y_1}{x_1} = \frac{\mathrm{tg}\,\lambda}{\sin\phi}\ ; \quad y_1^2 + x_1^2 = \frac{1}{\mathrm{tg}^2\phi + \sin^2\lambda}$$

ϕ the first time and λ the second time; the result, as equations of the fourth order (Lemniskaten): $\sin^2\lambda\,(x_1^2 + y_1^2)^2 = y_1^2 - x_1^2\,\mathrm{tg}^2\lambda$; $\mathrm{tg}^2\phi\,(x_1^2 + y_1^2)^2 = x_1^2 + y_1^2 \sin^2\phi$

The symmetry straight lines of the projection are $y_1 = 0$, i.e., the meridians $\lambda = 0$ and $\lambda = \pi$ on the one hand and $x_1 = 0$, i.e., the meridians $\lambda = -\frac{\pi}{2}$ and $\lambda = +\frac{\pi}{2}$ within the line segment $-1 < y < +1$ and the equator on the straight line $x_1 = 0$ outside of that line segment. The shape-true circle-grid which applied in the Littrow-Maurer map for the primary- and secondary-circles of the pair of points $\phi = 0$; $\lambda = \pm\frac{\pi}{2}$, i.e., $x = 0$; $y = \pm 1$, remains a shape-true circle-grid for the same pair of points in the new map also, since this point pair corresponds to itself

in the transformation through reciprocal rays. But even here, the property which prompted the including of SNo. 191 here, namely that the azimuth-equals with direction-points on the central-meridian are circles, would not be fulfilled if that circle-grid were the polar grid of the map. A map-grid of this projection with meridians and circles of latitude from degree to degree was calculated and sent along in 1931 with the "Graf Zeppelin" for its arctic trip as a radio positioning grid; it was published in 1933[89]. The same Lemniskaten grid (from 15° to 15°), represented by the dotted lines in figure 1, is a reproduction of the already mentioned work by von Wedemeyer[87] in 1918, if one considers the symmetry lines running from left to right as meridian $y = 0$ and the even numbers on it as polar intervals. The other symmetry line on the inner line segment means the meridians $\lambda = \pm 90^{\circ}$ with polar intervals to the equator noted; in the outer parts it means the equator itself, which is the branching-intersection of the Riemann Double Plane and on which filling out the noted numbers to 90° would yield the even numbers of the meridians λ.

Following is projection SNo. 192, proposed by F. August[90] in 1874. Its H and N are evolvents of epicycloids. It renders one of the best possible shape-true world maps, which lies completely within the finite and gives every point of the globe without exception as a shape-true point, even in appearance. The world map is derived as shape-true reproduction of the polar grid of a transverse-axical all-circular

projection (SNo. 170) with a central-H-length = 2 in such a manner that the infinite plane of the original map is reproduced in the interior of an epicycloid. This epicycloid describes a point of a circle of radius 1/2 if the circle rolls off on another one of radius 1. The latter circle in the all-circular map consists of the meridian $\lambda = \pm\frac{\pi}{2}$ and the poles are reproduced in themselves as the points y = 0 and x = ±1. If one calls Θ the angles between the central-meridian and the line joining the fixed and the rolling circle, then these equations apply for the epicycloid named, which gives the meridian $\lambda = \pm\pi$ as world map border:

$$x_1 = \frac{3}{2}\cos\Theta - \frac{1}{2}\cos 3\Theta$$

$$y_1 = \frac{3}{2}\sin\Theta - \frac{1}{2}\sin 3\Theta$$

or, in the form for the complex number levels,

$$u_1 = x_1 + iy_1 = \frac{3}{2}e^{i\Theta} - \frac{1}{2}e^{3i\Theta}$$

The equations for the all-circular map to be produced are:

$$z = x + iy; \qquad x = \frac{\sin\phi}{1 + \cos\lambda\cos\phi}$$

$$y = \frac{\sin\lambda\cos\phi}{1 + \cos\lambda\cos\phi}$$

On this map, $x = 0$; $y = \mathrm{tg}\,\frac{\pi}{2}$ applies for the equator; $y = 0$ applies for the meridian $\lambda = 0$; $x = \mathrm{tg}\,\frac{\phi}{2}$; and for the meridian $\lambda = \pm\pi$, $y = 0$, $x = \pm\cot\frac{\phi}{2}$.

For the transformation of the August Projection, one must

form $\frac{1 \pm \sqrt{1 - z^2}}{z}$, where $z = x + iy$. Then this complex equation applies for the transformed projection: $u = x_1 + iy_1 = \frac{p}{2}(3 - p^2)$. One finds for its equator $x_1 = 0$; $y_1 = \frac{1}{2}\left(tg\frac{\lambda}{4}\right)\left(3 + tg^2\frac{\lambda}{4}\right)$; for its meridian $\lambda = 0$ $y_1 = 0$; $x_1 = \frac{p}{2}(3 - p^2)$, where $p = \left(\cos\frac{\phi}{2} - \sqrt{\cos\phi}\right) : \sin\frac{\phi}{2}$; and for its border meridians $\lambda = \pm\pi$, $x_1 = \frac{3}{2}\cos 3\Theta - \frac{1}{2}\cos 3\Theta$; $y_1 = \frac{3}{2}\sin\Theta - \frac{1}{2}\sin 3\Theta$ where $\cos\Theta = tg\frac{\phi}{2}$. August referred to a relatively simple geometric construction for an arbitrary point (ϕ/λ) of his map.

The following comparison shows how fortunately this projection holds its own as a world map among the shape-true world maps which have the whole earth represented in a finite picture. If one assumes the length distortion in the center of the map $\phi = \lambda = 0$ to be equal to 1, then the August map suffers the greatest length distortion on the border meridian $\lambda = \pm\pi$, where it reaches the amount 4 for $\phi = 0$ and increases with increasing ϕ up to the amount 8 for $\phi = \pm\frac{\pi}{2}$. Of the shape-true circle-grids, however, those with the constant $n \geqq 1$ expand into infinity; and for those remaining finite, with constant $n < 1$, the length distortion becomes $\frac{n}{2}\sec^2\frac{n\pi}{2}$ at $\phi = 0$ on the meridian $\lambda = \pm\pi$ and increases to ∞ for $\frac{\pi}{2}$

SUBBRANCH B: SHAPE-TRUE; ORDER b: WITH NON-ALGEBRAIC GRID-LINES (SNo. 193, 194, 218)

The projection by Eisenlohr[91] (1870), SNo. 193, is a bit better than the August Projection, as far as length distortion is concerned, and August himself mentions this. The length

distortion in the Eisenlohr Projection has its highest value 5.83, everywhere the same on the entire world border meridian $\lambda = \pm\pi$, as compared with the value 1 at the point $\phi = \lambda = 0$. The equations for this projection read, in our complex way of designation:

$$y + ix = \frac{2}{1} (v - iu) + 2\sqrt{2} \left[e^{v-iu} - e^{-(v-iu)} \right] \qquad \text{where}$$

$$\operatorname{tg} u = \sin \frac{\phi}{2} : \left(\cos \frac{\phi}{2} + \cos \frac{\lambda}{2} \sqrt{2 \cos \phi} \right) ;$$

$$v = \frac{1}{2} \left[\frac{\cos \frac{\phi}{2} + \sqrt{\cos \phi} \cos \left(4 - 2 \right)}{\cos \frac{\phi}{2} + \sqrt{\cos \phi} \cos \left(4 + 2 \right)} \right]$$

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They are, however, so complicated, that they have never yet really been used for a practical production of a world map. Until further findings, the August Projection must be designated as the shape-true world map with the least area- and length-distortion. But it has, as Tissot-Hammer[92] state, not received the attention deserved. The work by Eisenlohr is not in Volume 71, as stated by August, but in Volume 72 of the Crell Journal.

Two further double-symmetrical, shape-true projections have been proposed independent of one another by C.S. Pierce (1879), SNo. 218, and by M.E. Guyou (1887), SNo. 194. They are simply related in that, when one map contains one of these projections as polar-grid, the other projection represents the grid of primary- and secondary-circles of the intersection-

point of central-meridian and equator. In other words, the Guyou Grid is the polar grid in transverse-axical inclination when the Peirce Grid applies as polar-grid in earth-axical inclination, and the other way around. This relationship does not seem to have been recognized before. It seems especially peculiar that the projection by the American Peirce[94] is not mentioned in the text book for map study of the *American Coast and Geodetic Survey* , but only on the closely related, just otherwise inclined, projection of the Frenchman Guyou[95] which is eight years younger. Guyou's projection belongs with the double-symmetrical projections of our family III, whereas that of Peirce shows different relationships of symmetry and therefore belongs in family IV. First, however, we want to treat the so-called *quincunctial projection* (five-form projection) by Peirce (SNo. 218), since it was the first to be proposed and can be somewhat more easily described.

If one designates the picture of the global point (δ/λ) on the complex map surface with $(x + iy)$, then Peirce employs $\operatorname{cosam}(x + iy + K;k) = \operatorname{tg}\frac{\delta}{2}(\cos\lambda + i\sin\lambda)$, where $k = \frac{1}{\sqrt{2}}$, and the angle of his modulus $= \frac{\pi}{4}$ and K is the elliptical integral of the first class,

$$\int_0^1 \frac{dz}{\sqrt{(1 - z^2)\left(1 - \frac{z^2}{2}\right)}}$$

The map picture, the classification of which is indicated in Illustration 14 (plate I), consists of infinitely many

congruent squares adjacent to each other on all sides, the periphery of which, such as A_1 A_2 A_3 A_4, is the picture of the equator, whereas square-centers alternately in both directions of the adjacency of the squares indicate the North Pole and the South Pole. Every four neighboring poles of the same designation (as S_1 S_2 S_3 S_4 and/or N_1 N_2 N_3 N_4), which are in the corners of a greater world map square, form a five-form--a quincunx--with the opposite pole (N_1 and/or S_1) in their center; thus the name of the projection. One may also call such a world map square a four-pointed star, the points of which make up the four-fold picture of the same global point. Other star-projections are the combination projections SNo. 228-231. Every semi-meridian $\pi = n \frac{\pi}{4}$ (n = whole number) between pole and equator on our map is a straight line; and these straight lines intersect at the pole at the proper angles. At the equator corners A, however, the apparent angle discrepancy increases up to 90^o.

The advantage of this projection is that it does not have a prescribed world map border. Full world maps in the above described star form are naturally only squares, the four corners of which are pictures of the same earth pole. On the other hand, that quadrilateral is also a full world map, whose straight parallel laterals, such as BE or CD, represent the half equator and are separated from each other by one fourth of a length of the equator ($=A_1$ A_3), whereas the two other laterals can be arbitrary curves, BC and EB, which must only

be congruent and parallel. The arrangement of the globe eighths lined up beside each other can be seen from the reproduction; each of the globe eighths appears as an iscoceles-rectangular triangle. The four parts of the northern hemisphere are disignated with 1,2,3,4, and those of the southern hemisphere with I, II, III, IV. The dotted curves correspond approximately to the meridians $\lambda = 15^{\circ}$ and $\lambda = 195^{\circ}$. While the squares $[S_1\ S_2\ S_3\ S_4]$, $[N_1\ N_2\ N_3\ N_4]$, $[N_1\ N_4\ N_5\ N_6]$, $[N_1\ N_6\ N_7\ N_8]$, $[N_1\ N_8\ N_9\ N_2]$ are full world maps, the equally great squares with corner points A are not. On the other hand, not only rectangles, such as $[A_9\ A_{10} A_2\ A_1]$, $[A_3\ A_8\ A_5\ A_1]$, $[A_4\ A_2\ A_6\ A_{11}]$ and $[A_3\ A_2\ A_{11} A_{12}]$ are full world maps, but also the already mentioned quadrilateral B C D E. Although overall true-shape is proven in Peirce's projection, when subjected, even in the corners of the equator square, to microscopic mathematical examination, a world picture with square equator really appears macroscopically quite unsatisfactory, no matter if a hemisphere is torn up into four three-cornered lobes in the star-form such as $S_1\ S_2\ S_3\ S_4$, or if both hemispheres cohere at only one quarter of the equator in the rectangle such as $A_3\ A_5\ A_6\ A_4$.

The following must be noted with respect to the family to which Peirce's grid should belong: It is not possible to give such a closed world map such a delimitation that the grid shows the double-symmetry characteristic of family III, since the equator does not form a uniform straight line as symmetry-line of the grid, even though it is symmetrical to each of its four

straight-line parts. We must assign the projection to family IV. This is another matter if we consider the straight-line full meridian S_1 N_1 S_3 in Peirce's map as the great secondary-circle and go over to the grid of primary-and secondary-lines, i.e., to Guyou's projection. This grid is doubly-symmetrical and belongs to family III. It, too, is a quincuncial grid which fills out the entire map surface with squares, each representing a hemisphere, but now an east- or west-hemisphere, rather than a north- or south-hemisphere. The particular world map is again either a rectangle with lateral ratio of 1:2, as FGHJ in Illustration 14 (plate I), in which A_2 A_3 represent two-fold the north pole A_1 A_2 and two-fold the south pole and the two hemispheres connect only at one piece of meridian of 90° arc length; or a square star with a square east- or west-hemisphere in the center and another hemisphere slit up into four triangular lobes. Illustration 15 (plate V) shows a part of such a world map with 270° longitudinal difference on the straight-line equator from the north pole--reproduced as two points N--to 30° south. The ±90° longitude passes from the equator to 45° latitude as a perpendicular at the equator and is then split into two parts parallel to the equator. At the slit-points, the circle of latitude 45° appears rectangularly broken. The four quarters of the globe between meridians are designated as I, II, III, IV, whereby the arrangement of contiguous world maps is made visible. The dotted line shows how some lines in such a world map assume unnatural forms--the picture of the globe-circle of 60 arc degrees about the central

point M ($\phi = \lambda = 30°$). Nevertheless, a world map according to this projection looks quite acceptable, when one can lay the rectangular meridian break in the middle of the ocean, as is done on the map in the American textbook[92]).

The mathematical derivation given by Guyou for his projection is very singular. For position-finding on the globe, he uses two hosts of "globe ellipses" which intersect themselves rectangularly. Such an elliptical line of latitude is the entirety of all points whose arc intervals on the north- (south-) hemisphere yield a constant sum from the two fixed points $\lambda = \pm 90°$; $\phi = 45°$ north (south). The curves rectangular to this host of elliptical lines of latitude are the "elliptical meridians". Each one of these connects all points whose arc intervals on the east- (west-) hemisphere yield constant sums from the two fixed points $\phi = \pm 45°$ and $\lambda = 90°$ east (west). That latitude ϕ applies as elliptical latitude coordinate ϕ_0 of a latitude ellipse at which this globe ellipse intersects the zero meridian $\lambda = 0$. That longitude λ, at which the concerned globe ellipse intersects the equator, applies as elliptical longitude coordinate λ_0 of a meridian ellipse. The equations which apply between these different coordinates on the globe are:

$$\sin \phi = \sin \phi_0 \sqrt{1 - \tfrac{1}{2} \sin^2 \lambda_0}; \quad \cos \phi \sin \lambda = \sin \lambda_0 \sqrt{1 - \tfrac{1}{2} \sin^2 \phi_0}$$

Guyou reproduces the grid of elliptical latitude- and longitude-lines of the globe now as grid of straight lines x = const and y = const, shape-true on the map surface, the equator x = 0 and the central-meridian y = 0, according to the reproduction-

equations:

$$x = \int_0^{\phi_e} \frac{d\phi_e}{\sqrt{1 - \frac{1}{2}\sin^2\phi_e}} \; ; \qquad y = \int_0^{\lambda_e} \frac{d\lambda_e}{\sqrt{1 - \frac{1}{2}\sin^2\lambda_e}}$$

(Guyou uses other letters in his paper

Our designations:	x	y	ϕ	λ	ϕe	λe
Guyou's designations:	y	x	L	G	λ	γ

)

The shape-true projections of Peirce and Guyou, with *quadratic* hemisphere grid are nevertheless the most advantageous special cases of the more general of such shape-true, double-periodical projections with *rectangular* grid, where each rectangle represents a hemisphere. The equations of Guyou may be generalized in the forms:

$$\sin\phi = \sin\phi_e \sqrt{1 - \cos^2 F \sin^2\lambda_e}$$

$$\cos\phi \sin\lambda = \sin\lambda_e \sqrt{1 - \sin^2 F \sin^2\phi_e}$$

$$x = K \int_0^{\phi_e} \frac{d\phi_e}{\sqrt{1 - \sin^2 F \sin^2\phi_e}} \; ;$$

$$y = K \int_0^{\lambda_e} \frac{d\lambda_e}{\sqrt{1 - \cos^2 F \sin^2\lambda_e}}$$

where K is a fixed scale value and the focal points of the groups of globe ellipses are $\lambda = \pm \frac{\pi}{2}$ $\phi = \pm F$. Guyou, who pointed this out, chose $K = 1$, $F = \frac{\pi}{4}$ for his projection.

FAMILY IV: LESS REGULAR; NEITHER TRUE-CIRCULAR NOR DOUBLE-SYMMETRICAL

(No. 197-225)

The projections of the last subbranch in family III, in which the H are parallel straight lines, will be mentioned later in connection with family IV, to which we now turn our attention. In connection with SNo. 92, we have already mentioned SNo. 197 and 198 of this family, the cylinder-perspective projections which are no longer double-symmetrical if the sighting point leaves the center of the globe; also mentioned in the note on page 67 were the cylinder-circular projections SNo. 208-214, which are produced when the determining equations, really valid only for a hemisphere with respect to projections No. 104-106, 113-115, 124-126, 132, 134 are applied for an entire world map. We have also already mentioned the cylinder-radial interval- or area-true projections of family IV, which are produced from the double-symmetrical cylindrical projections SNo. 87-91 when their N-lines are substituted for with parallel curves. No. 201-204 are such projections, in which the parallel curves are circle-arcs with radius r. No. 201, the interval-true projection on a contiguous-cylinder, has the equations $y = \lambda$; $x = \phi + r - \sqrt{r^2 - \lambda^2}$ The entire arc length of each N is $L = 2r\alpha$, where $\sin \alpha = \pi : r$. Here L becomes always greater than the basic-circle 2π on the globe, since $r > \pi$. True-shape exists then only at point $\phi = \lambda = 0$.

The equations for SNo. 202, the interval-true projection

on an intersecting-cylinder, are:

$$y = n\lambda; \quad x = \phi + r - \sqrt{r^2 - n^2\lambda^2}; \qquad n < 1;\ r > n\pi$$

Now the entire length of each N becomes $L = 2r\alpha$, where $\sin\alpha = n\pi : r$ $2r\alpha$ can then become equal to the entire length of any two N $\pm\phi_m$ on the globe, and true-shape is produced at the two points $\lambda = 0$ and $\phi = \pm\phi_m$. Example:

$n = 1 : \sqrt{2}$, $r = \pi$, $\alpha = 45°$, $\cos\phi_m = \pi : 4$; $\phi_m = \pm 38.2°$.

Shape-true points, $\lambda = 0$ and $\phi = \pm 38.2$.

$y = \lambda$; $x = \sin\phi + r - \sqrt{r^2 - \lambda^2}$ apply for the area-true projection on a contiguous-cylinder (SNo. 203). Arc length of the N and true-shape are as for No. 201. $y = n\lambda$; $x = [\sin\phi + r + \sqrt{r^2 - \lambda^2}] : n$ apply for the area-true projection on intersecting-cylinder (No. 204). For arc length of the N the same applies as for No. 202. There is true-shape, however, only for $\lambda = 0$ and $\phi = \pm\phi_w$ where $\phi_w = n$. Example:

$n = 1 : \sqrt{2}$; $r = \pi$; $\alpha = 45°$; $\phi_m = \pm 38.2°$, but $\phi_w = \pm 45°$.

If one rejects the equidistance of the cylinder-rays H, then one can make the N-circle-arcs equal-spaced. SNo. 205 and 206 are such projections. The first is interval-true, the other zone-true.

Special Stipulations for Projections

In order not to go too far afield in discovering further projections, it is expedient to refer to practical stipulations which lend the map valuable properties or facilitate, by means of the grid, the solution of geographic or navigation problems on the earth or astronomical problems in the sky. We treat this question in a general manner and state which projections fulfilling such stipulations have already been named in families I, II, and III, and which of the ones with less grid regularity should yet be assigned to family IV. The stipulations concerned here refer either to the reproduction of angles, areas or lines which are of importance on the globe for geographic or nautical (astronomical) purposes. We consider the global great-circles (orthodromes) as the shortest (geodetic) lines of the globe, the course-equals (loxodromes) as lines of constant course on the globe, and the azimuth-equals as locating-lines of those points from which a direction-point appears in constant azimuth.

A. Stipulations Concerning the Angles (SNo. 217, 220)

The stipulation for true-shape in the entire map is given by means of differential equations, to which we have referred on page 67 and which make extremely diverse grid-lines possible, so that such projections appear in all families. The only shape-true radial projections we find in family I are the all-circular (No. 2) and the conical maps (No. 39 and 40); in family

II, the Etzlaub-Mercator Maps No. 94 and 95; in family III, the row-circular shape-true circle-grids of the first class, No. 169-173, and of the second class, No. 181, as well as the non-row-circular No. 189-194 of Littrow-Maurer, Fiorini, August, Eisenlohr and Guyou. As belonging to family IV we consider the projection by Peirce (SNo. 218), which does without the double-symmetry and the shape-true circle-grids after Lambert and Lagrange, in which not the equator, but another circle of latitude is reproduced as a straight line and which are therefore not symmetrical to the central-H; plus the lateral world maps of such shape-true circle-grids, which inversely are only symmetrical to the equator. The first have been included as SNo. 217, the second as SNo. 220, in family IV.

True-shape at only two points, or at only one, is only stipulated when the great circle is a straight line through the shape-true point; more about this under C below.

B. Stipulation for True-area (SNo. 203, 204, 212, 213)

This stipulation, too, designated by the differential equation

$$\frac{\delta x}{\delta \phi} \frac{\delta y}{\delta \lambda} = \frac{\delta x}{\delta \lambda} \frac{\delta y}{\delta \phi} = \cos \phi$$

can be fulfilled in manifold forms of the grid-lines, so that area-true projections appear in all the families. In family I we find the radial point-maps by Lambert (SNo. 23), the radial ring-maps No. 28 and 29 and the conical projections No. 51

and 52 (Albers). Family II offers the true-cylindrical maps No. 87 by Lambert, No. 89 by Behrmann and the extremely numberous untrue-cylindrical SNo. 131-149, among which are ones by Collignon, Mollweide, Nell, Hammer, Wagner, Crastner, Prepetit-Foucaut and Bourdin. In family III we find the all-conical projection SNo. 179, the transformed projections No. 183, 184, 186; and even among the combination projections of family V, we shall discover true-area. We have already pointed out the area-true maps of family IV (SNo. 203, 204, 212, 213). For more concerning an area-true counter-azimuthal projection (SNo. 219), which likewise belongs in general to family IV, see E2.

C1. Straight Lines as Pictures of Great Circles of the Globe (SNo. 199, 195)

Maps in which *all straight lines* are pictures of great circles are called straight-directional or orthodromic. Such are the central-perspective projection (SNo. 1) and its collinear reproductions, also Field No. 81 on page 182. They contain in general two shape-true points (SNo. 182 in family III), whereas the special case (SNo. 1) indicates greater regularity of central-circularity with only one shape-true point. The term "gnomonic" is to be limited to only this special case.

The stipulation that the great-circles of a global point be reproduced as straight lines which intersect themselves at proper angles is called *radial-directionality* (Radstrahligkeit) (sometimes also called cluster-radiality (Buschelstrahligkeit)).

If one stipulates *radial-directionality for two points* then the *double-radial-directional* projection is therewith fully determined, including the scale, and it is in this manner that one obtains the straight-directional projection SNo. 182.

A projection has been proposed, which is *radial-directional* at only *one point S*, whereas not radial-directionality, but cylinder-radiality is stipulated for a second point, the earth pole, that is, the meridians should be equidistant parallels. This suggestion (SNo. 199) was offered by Schoy[96]. This projection, which is *radial-* and *cylinder-radial*, has been incorrectly designated as azimuthal, which does not take into account that the central-circularity necessary for this is missing.[97] His equations read: $y = \lambda$; $x = \lambda[\cos\phi_0 \operatorname{tg}\phi - \sin\phi_0 \cos\lambda]:\sin\lambda$

It is to be noted here that one can combine the cylinder-radial reproduction of the great circle which is perpendicular to the central meridian with cylinder-radial reproduction of the meridians, thus achieving a *double-cylinder-radial* projection. This corresponds with a naive attempt to simply consider the globe as a plane. Its equations would be $y = \lambda$ $x = \psi$, where ψ is the angle which each great circle of the second group forms with the equator, thus $\cot\psi = \cot\phi \cos\lambda$ To be sure, the grid in the vicinity of the point $\phi = \lambda = 0$ is very advantageous and exhibits scale-true equator and central-meridian. But it is extremely disadvantageous for areas in the vicinity of $\lambda = \pm\frac{\pi}{2}$. Each of these two meridians shrinks

together into a pair of points $y = \pm\frac{\pi}{2}$; $x = \pm\frac{\pi}{2}$ whereas the entire line segments $\frac{\pi}{2} < x < \frac{\pi}{2}$ on the lines $y = \pm\frac{\pi}{2}$ are the pictures of the points ($\phi = 0$; $\lambda = \pm\frac{\pi}{2}$) According to its symmetry relationships, this projection (SNo. 195) belongs to family III.

C2. Circles as Pictures of Globe Great Circles

Maps which reproduce all great circles of the globe as circles are designated as *great-circle-true*. For the radial one among them, the radius rule stated by Schols and already stated on page 25, applies:

$$r = 2\ \mathrm{tg}\ \delta : (1 + \sqrt{1 + k^2 \mathrm{tg}^2 \delta}) = 2 : [\cot \delta + \sqrt{k^2 + \cos^2 \delta}]$$

From among these radial great-circle-true projections, the central-perspective SNo. 1, with the constant $k = 0$, deserves special attention, since it covers the surface of the map two times, with one global great circle represented as infinitely distant straight lines of the picture plane and all great circles as straight lines. For $k > 0$, the radial-circular map covers the picture plane only once and represents the point $\delta = \pi$ as infinitely distant straight lines, whereas the circle $\delta = \frac{\pi}{2}$ obtains the radius $2:k$ in the picture.

The following example shows how bizarre a form a reproduction

can assume, which fulfills the stipulation for being great-circle-true, a sensible stipulation per se: A straight-directional map with two shape-true points, A and B, is great-circle-true, since all great circles on it are straight lines. We reproduce this map by means of reciprocal rays from point M_1 which is the picture of the mid-point of the arc AB on the globe. In the resulting picture all the great circles are represented by circles. The point M of the globe is now represented as the inifinitely distant straight lines, and the entire global great circlè, the middle of which is the point M of the globe, is represnted now by a point 0. All great circles of the globe are represented by the cluster of all circles passing through 0; and the global great circles which pass through the global point A and/or B are the circle-cluster through 0 and the picture-point A_1 of A in the picture, and/or the circle-cluster through 0 and the picture-point B_1 of B, while true-shape exists at the same time at points A_1 and B_1.

Dl. Straight Lines as Pictures of Course-Equals (Loxodromes) (SNo. 207, 224, 225)

The stipulation for reproducing all course-equals as straight lines is fulfilled most usefully for navigation by the true-shape marine map according to Etzlabu-Mercator (SNo. 94 and 95). Aside from that one, every collinear reproduction of the Mercator map represents the course-equals as straight lines. True-shape is generally lost with perspective reproduction.

However, the map remains yet a cellular (zenithal) prime-point-map, if the point of vision lies infinitely distant. Then, the map also remains cylindrical, if the picture plane is parallel to the central meridian or to the equator of the map being reproduced, as with SNo. 96; on the other hand, with any other position of the picture-plane, the parallel straight-line meridians lie oblique to the equator (SNo. 224). If the point of vision lies in infinity, then the map is neither a prime-point map nor cellular, but remains cylinder-circular and secondary-circle-spaced, if the picture plane stands parallel to the equator of the map to be reproduced (SNo. 207), whereas these two properties are also lost with another position of the picture plane (SNo. 225). All these projections have no practical significance; however, Maurer was able to more clearly present properties of the course-equal projections in one grid after No. 96, about which ambiguity had existed in the literature.

D2. Circles as Pictures of Course-Equals (SNo. 217, 220-223)

One obtains a map in which all course-equals of the globe are represented as circles, if one reproduces by means of reciprocal-rays one of the projections named under D1 with straight course-equals. Then, true-shape is maintained, if the map being transformed is a Mercator map. On a projection obtained in this manner, the north and south poles converge at one point in infinity, so that such a grid can be used for

examining the angle relationships at the poles, which are not accessible on the Mercator map. As a matter of fact, proof has been given with such a grid that the Mercator map is shape-true at the poles also, although the appearance of the parallel meridians contradicts this[28]. Such a shape-true map with circle shaped course-equals is at the most simple-symmetrical, for example, with the equator as symmetry-line when the point of intersection of the reciprocal rays lay on the equator (SNo. 222), or fully unsymmetrical, when this point of intersection lay neither on the equator nor on the central-meridian (SNo. 223).

One can consider a map like SNo. 222 as a special case of one of the before mentioned lateral maps with shape-true circle-grids, which are in our system as SNo. 220, namely, as that special case where the entire center of the map shrinks together to one point. Also the general case of a lateral map with shape-true circle-grid represents an only simple symmetrical grid (SNo. 221) of family IV, if the central map is not a point and north and south poles thus lie apart from each other, and if another circle of latitude, rather than the equator, is reproduced as straight line (SNo. 217). There are also certain course-equals in the shape-true circle-grids SNo. 169-173 and 217, as in their lateral maps No. 220 and 221, namely, the meridians and circles of latitude as course-equals of the course angles 0° and 90°, which are reproduced as circles. But no other course-equal in them is a circle, since every other course-equal must pass through both poles, all circles in those

maps, however, representing meridians through both poles.

E1. Straight Lines as Pictures of Azimuthequals (SNo. 200, 205, 216, 219, 196)

As has already been mentioned, not all ∞^3 azimuthequals of the globe can be represented with only ∞^2 straight lines of the flat surface. But all straight lines of the plane of the picture can certainly be pictures of azimuth-equals on an equal-azimuth map. The purpose of such a map can only be that the parallel azimuth of every azimuth-equal of the map be able to be read directly as intersection-angle of this azimuth-equal with the meridian of the direction-point. It must be stipulated that the ∞^1 direction-points of the straight ∞^2 azimuth-equals lie on a meridian represented by a straight line along which there is true-shape. The derivation for the projection (SNo. 189), which accomplishes this, was stated in 1905 by Maurer. Naturally, all straight lines of the picture surface remain pictures of the same bundle of ∞^2 azimuth-equals in every collinear reproduction of this projection. But since true-shape cannot be maintained along the meridian of the direction-point in such a reproduction, every such azimuth-equal map would miss its purpose.

Projections, which reproduce the azimuth-equals of a direction-point as straight lines intersecting at the proper angle, were designated as counter-azimuthal in 1910 by Hammer. For the sake of brevity, such a direction of a counter-azimuthal map is to be

called a *counter-azimuthal point*. The projection by Littrow-Maurer (1883 and/or 1905) thus contains ∞^1 counter-azimuthal points and is chronologically the first counter-azimuthal projection. The first to have only one single counter-azimuthal point (Mecca) was offered in 1909 by Craig[99] as *retro-azimuthal*, with the added stipulation that the meridians be equidistant parallels. This is thus a *cylinder-radial counter-azimuthal map*. If the counter-azimuthal point, as in Craig's projection, does not lie on the equator, then the projection is only simply-symmetrical (family IV); it comes into family III for $\phi_0 = 0$. We include it, then, as SNo. 200 in our system. Its equations are

$$y = \lambda; \quad x = \lambda\ [\sin\phi \cos\lambda - \cos\phi\ \mathrm{tg}\ \phi_0]:\sin\lambda$$

It was shown for the center-interval-true counter-azimuthal projection by Hammer (SNo. 78) that the stipulation of a counter-azimuthal point $\lambda=0;\ \phi=\phi_0$ yields the determining equations $x = y \cot A;\ \cot A = \mathrm{tg}\ \phi_0 \cos\phi\ \ \mathrm{cosec}\ \lambda\ \ -\sin\phi \cos\lambda$, where A is the constant azimuth. $y = \lambda$ is employed by Craig as condition for cylinder-radiality, in place of the stipulation of Hammer for center-equidistance $x^2 + y^2 = \delta^2$, where

$$\cos\delta = \sin\phi \sin\phi_0 + \cos\phi \cos\phi_0 \cos\lambda$$

The distribution of the circle of latitude on each meridian λ obtains the form: $x = a \cos\phi - b \sin\phi$, where $a = \lambda\ \mathrm{cosec}\ \lambda\, \mathrm{tg}\ \phi_0$ and $b = \lambda \cos\lambda$

Zoppritz-Bludau[100] include a Craig map with Berlin as direction-point.

The central-circularity of the counter-azimuthal projections SNo. 78, 79, 80 is lost with the Craig stipulation for cylinder-radiality. The result is the same if we expect true-area from a counter-azimuthal projection. In order to carry that out, one would have to reproduce the circles δ= const as such curves that the sector between such a piece of curve and each two straight azimuth-equals of the counter-azimuthal point would be equal to the piece of area that is bordered on the globe by the two azimuth-equals and the circle δ = const--a very complicated job which is hardly worth the trouble. The projection (SNo. 219) has only been included in the system in order to show that true-area and counter-azimuthality can be combined. The projection belongs in general to family IV,but it becomes double-symmetrical (family III) if the counter-azimuthal point falls on the equator.

We find counter-azimuthal maps in family I as true-circular prime-point-maps with or without equidistance (SNo. 78-80). As cylindrical ring maps, with or without equidistance, they belong to family II, branch B in the special case $\phi_0 = 0$, whereas the two forms (SNo. 215 and 216) in the general case $\phi_0 \gtrless 0$ belong to family IV.

The *double-counter-azimuthal* projection by Maurer[101] in 1919 (SNo. 196) belongs yet to the counter-azimuthal maps of

family III. The two counter-azimuthal points lie on the equator $\phi = 0$; $\lambda = \pm\lambda_1$. The equations of the projection are:

$$x = 2\sin\phi\,(\cos^2\lambda_1 - \sin^2\lambda):\sin 2\lambda_1 ; y = \sin 2\lambda:\sin 2\lambda_1$$

The grid (Illustration 16, plate VII) is double-symmetrical. The meridians are parallel but not equidistant, straight lines. The circles of latitude are ellipses which all pass through the points $x = 0$; $y = \pm\lambda_1$. The poles are represented by the two circles $N_1\ N_2\ N_3\ N_4$ and $S_1\ S_2\ S_3\ S_4$. In the reproduction, the two direction-points A and B are put at $\phi=0°$, $\lambda = 50°W$ and $\phi = 0°$, $\lambda = 80°W$, that is, just about where the equator meets the coasts of South America. The cell between the meridians 20°W and 110°W lies inside the border $N_1\ N_2\ S_2\ S_1\ S_4\ N_4\ N_1$, in which the borders of the continents are also indicated with dotted lines. The adjacent global cells between the meridians 20°W to 10°E and 110°W to 140°W are formed in the two wedges N_2 A S_2 and N_4 B S_4, so that the entire meridians 10°E and/or 140°W shrink together into the points A and/or B. The rest of the global cells between the meridians 10°E and 25°E and/or between 140°W and 155°W fall in the wedges A N_3S_3 and/or B N_3S_3, by which the one hemisphere is reproduced. Furthermore, every earth point is congruent with its antipodal point. The semi-ellipse circle-of latitude-pictures cutting through the equator are not drawn, since they squeeze together too much. It is interesting to note that this map with two counter-azimuthal points lies entirely with the finite, whereas the map by Craig with only one such point, as well as the aziruth-equal map with infinitely many counter-azimuthal points, reach into infinity.

FAMILY V: COMBINATION GRIDS (SNo. 226 - 237)

Branch A: Double-symmetrical (SNo. 226,227)

Whereas a uniform mathematical expression for the coordinates x and y or for the equation for each H or for each N applied for the whole world map in the projections of families I through IV, the combination grids of family V exhibit different forms of the reproduction rule for the particular parts of the map.

In branch A are presented doubly-symmetrical projections in which the same law applies neither for the N (Branch A) nor for the H (Branch B) on the whole map. Pictures of branch A produce, as already mentioned on p.67, those doubly-symmetrical projections in which every meridian is represented as a pair of straight lines symmetrical to the equator. They are all brought together in family V under SNo. 226. The rule for the N in them does apply for the whole map.

As example of a projection in which the rule for the H applies for the whole map, while each N consists of four arcs of different radius and without break, we may use Maurer's *Isogon Map* of 1911 (SNo. 227). The central part of the map (Illustration 17, plate I) pictures the eastern hemisphere

with the map center O ($\phi = 0$; $\lambda = 90^{\circ}$) and the radius OP = 1 in the transverse all-circular projection, its H-line-rule

$$x^2 + \left| y - tg \left(\frac{\pi}{4} + \frac{\lambda}{2} \right) \right|^2 = sec^2 \left(\frac{\pi}{4} - \frac{\lambda}{2} \right)$$

also applying for the western hemisphere, so that all H-circle-lines can be drawn. The equations

$$(x - cosec\ \phi)^2 + y^2 = cot^2\phi = r^2_1$$

apply for the N-arcs inside the eastern hemisphere and its radius r_1.

If the arc $N_2N_0N_1$ of the inner hemisphere is drawn about its center M ($y = 0$; $x = cosec\ \phi$), then it is first lengthened by the two arcs N_1N_3 and N_2N_4 with the radii

$$r_2 = tg \left(\frac{\pi}{4} - \frac{\phi}{2} \right)$$

the centers of which are the intersection points D_1 and/or D_2 or the polar tangents D_1OD_2 with the rays MN_1 and/or MN_2 and further about the semi-circle about center P with radius $PN_3 = PN_4 = r_3 = 2\ tg \left(\frac{\pi}{4} - \frac{\phi}{2} \right)$.

Illustration 18 (plate IX) is a redrawing of the map of 1911, in which the directions of the isogons is from the year 1931. Aside from the strongly distorted South America, the

shapes are quite good. Above all, however, such a map is much more appropriate than the usual isogon map in the Mercator projection for clarifying the general course of the isogons. Every magnetic-declination-equal passes through the two earth poles as well as through the two magnetic poles of the earth, all four of which lie here in advantageous areas of the map, whereas the Mercator map does not give the earth poles and mostly only one magnetic pole.

BRANCH B, SUBBRANCH A: TRUE-CIRCULAR (SNo. 228 -231)

In branch b, the projections are not *doubly-symmetrical*, because the equator is not represented as a straight line. In the first subbranch A we find projections, too, in which the law for the N applies for the whole map; this rule is $r = f(\delta)$ as for the true-circular projections in family I. But no uniform rule applies for the H.

We find the *star-projections* in the first order of the subbranch, in which the northern hemisphere is presented as a straight-cellular reproduction as according to family I, the southern hemisphere, however, as slit up into n prongs along certain border meridians, each of which ends up at a point representing the south pole. The three projections SNo. 228,

229, and 230 are all center-interval-true ($r = \delta$) and reproduce the H of the southern hemisphere as straight lines. The northern hemisphere of No. 228 and 229 is radial; the number of lobes of the southern hemisphere is 8 in Petermann's Star Projection, 5 in that of Berghaus. The lobes in Petermann's projection are not uniform with the border meridians 10°, 60°, 100°, 155°E and 35°, 85°, 120°, 155° W, in order to avoid distortions of Africa, Australia, America. In Berghaus' projection the lobes are congruent with the border-meridians 56°, 128°E and 16°, 88°, 160° W, so that Australia is cut through the middle. One comes out better with six uniform lobes, as is shown in SNo. 231 and 233.

SNo. 230, Steinhauser's *conoalatic* (konoalatischer) projection has a conical northern hemisphere and four lobes congruent with the border meridians 0°, $\pm 90^\circ$ and 180°. Since the map is projected in a sector of 240°, its north pole becomes a point, if it is projected center-interval-true on an intersecting cone through $\delta_m = 85.7^\circ$ ($3 \sin\delta_m = 2\delta_m$) and reproduces δ_m true to scale. The newly added star projection SNo. 231 has a northern hemisphere in area-true radial projection with the radius $r_1 = 2 \sin \frac{\delta}{2}$. For the six lobes of the southern hemisphere congruent with the border meridians 40°, 100°, 160° E, 20°, 80°, 140° W ,

the radius rule is the radius rule mirrored at the equator of the northern hemisphere, thus,

$$r_a = 2\left(\sqrt{2} - \cos\frac{\delta}{2}\right)$$

Then, true-area is also produced for the southern hemisphere, if we make the uniformly distributable entire length of each of its circles of latitude just as great as that of the circle of latitude in the comparable northern latitude. If one counts λ from the central meridian of the lobe represented in straight line in a lobe of the southern hemisphere $\left(\delta \geqq \frac{\pi}{2}\right)$ then these equations apply:

$$r_a = 2\left(\sqrt{2} - \cos\frac{\delta}{2}\right) \text{ and } \quad = r_i : r_a \quad \text{where}$$

$$r_i = 2 \sin\frac{\pi - \delta}{2} = 2 \cos\frac{\delta}{2}$$

Illustration 19 (plate VIII) shows this very pleasing form of the grid with slightly curved H of the southern hemisphere.

BRANCH B, SUBBRANCH B: NEITHER DOUBLY-SYMMETRICAL NOR TRUE-CIRCULAR

(SNo. 232 - 237)

In Subbranch B we find no uniform law for the whole map, neither for the H nor for the N. Here, too, the first order offers *star-projections*, where, however, the equator is not a

uniform curve but is an n-corner in the first class. Each side of an n-corner is symmetry-straight for the pictures of the halves of that global-cell of which a piece of the equator is the original side of an n-corner. The border meridians, as well as the central meridian of each such half of a global cell, that is, of each lobe of the southern hemisphere, are straight lines. In the subclass, each H is a pair of straight lines, and each N of a lobe is a straight line parallel to the piece of the equator of the lobe. The H can be equally or unequally spaced, the n-corners regular or not regular. In SNo. 232, we find an irregular octagon with equal-spaced H at the same border meridians as in No. 228. Jager[103] presented this projection in 1865. Illustration 20 (plate VII) shows the irregular star, the southern tips of which stand off at varying distances from the center of the star, when the central-H of the lobes are spaced scale-true, distortion of the lands of the earth nevertheless being avoided. One could have achieved the latter also with a regular six-lobed star, and thereby have had the advantage, that the equator and the six border meridians (the same as in No. 231) could be reproduced true to scale (SNo. 233). Illustration 21 (plate VIII) shows the extremely simple grid, which is so obvious that it has probably been used here and there.

It is quite simple to attain an *area-true* projection (SNo. 234) in this subclass; as example of this we chose a quincunx representation with uniform equator form and polar inclination as in the projection by Peirce (SNo. 218). The equator and each circle of latitude of the northern hemisphere becomes a square. One global triangle between the north pole and the equator-points $\phi = 0$, $\lambda = \pm\frac{\pi}{4}$ is reproduced in an isosceles-rectangular triangle with height and half hypotenuse $y = x = \sqrt{\pi}\sin\frac{\delta}{2}$, and one need only space the N-lines, the hypotenuses of these triangles, uniformly, in order to obtain an area-true representation in a quincunx grid. When $x = 0$ of the beginning meridian $\lambda = 0$, the equations which apply for the eighth of the globe in the region $-\frac{\pi}{4} < \lambda < \frac{\pi}{4}$ and $\delta < \frac{\pi}{2}$ are:

$$y = \sqrt{\pi}\sin\frac{\delta}{2}\ ; \quad x = \left(4\lambda \sin\frac{\delta}{2}\right) : \sqrt{\pi}$$

The grids in each two adjacent eighths of the globe are symmetrical to their common side.

The rectangular breaks of the N in a grid such as the projection No. 234 are especially irritating. One may avoid them, if one rounds off the corners, e.g., through quadrants. If N in Illustration 22 (plate I) represents the pole and each of the square sides AB and BC represent a quarter of the equator, then for area-equality of the globe octant NABC, the quadrant

side must be $\sqrt{\frac{\pi}{2}}$. Now one could draw for C and a each a quadrant through B and N and let the quarter of the N-line δ consist of the two straight line segments DE and GH and the quadrant EFG about the center J. If one then takes the coordinates of the point E, ND = y; DE = x, that is, the circle radius JE = y - x, then the stipulation for true-area between the area NDEFGH and its original on the globe yields the equation

$$\pi \sin^2 \frac{\delta}{2} = 2xy - x^2 + (y - x)^2 \frac{\pi}{4}$$

and, between x and y, the equation for the circle NEB:

$$y^2 + \left(\sqrt{\frac{\pi}{2}} - x\right)^2 = \frac{\pi}{2}$$

x and y are given in their dependence on   through both equations, so that the N can be drawn. The following little table gives correlated values for the case of a uniform globe of 40 mm radius.

[See table on page 167a following.]

It is quite complicated to space the thus area-equal zones exactly area-true through the H. Illustration 23 (plate VIII) gives the grid of such a star map of 15° to 15° in length and width. The arcs are drawn dotted on an octant of the globe; the straight and arcing segments of the N converge on them. The H are simply drawn as pairs of straight lines which divide the equator uniformly. The deviations from complete true-area are thus only slight, and the area-equality of the zones is

δ	0°	15°	30°	45°	60°	75°	90°
y (mm)	0	10,03	20,00	28,80	36,73	43,72	50,13
x (mm)	0	1,02	4,16	9,10	16,05	27,30	50,13

exactly preserved in this projection SNo. 235. If x_o and y_o are connected in this projection according to the equations

$$\pi \sin^2\frac{\delta}{2} = 2x_0y_0 - x_0^2 + (y_0 - x_0)^2\frac{\pi}{4};$$

$$y_0^2 + \left(\sqrt{\frac{\pi}{2}} - x_0\right)^2, = \frac{\pi}{2}$$

in the first quarter of the northern hemisphere, we find the equation for N: $y = y_o$ so long as x ` x_o and

$$(x - x_0)^2 + (y - y_0)^2 = (y_0 - x_0)^2$$

for $x > x_o$, the equation for the H-straight lines:

$$tg\ \alpha = 4\lambda : \pi$$

For the last order of our system, the *many-surface- projections (Polyhedral projections),* one imagines the uniform globe as surrounded by a polyhedron, with a piece of the globe represented on each of its surfaces. Then the polyhedron is to be cut up, adjacent surfaces laid next to each other and so spread out for a map surface. Insofar as one expects to produce an uninterrupted, coherent world map, the reproduction rule must make sure that that same line from the globe is reproduced exactly the same where an edge is formed by two adjacent polyhedron surfaces. This is not the case with the so-called

"Prussian Polyhedral Projection"[104] nor with the *"modified polyconical projection"* proposed by Lallemand[105] for the international world map with scale of 1 : 1,000,000. We shall leave such manners of reproduction our of our system, since they actually give no uniform reproduction of the whole globe on a plane surface. They are really reproduction of small pieces of the globe on *very many single leaves* only a few of which can be considered as adjacent to each other at given times, as long as the resulting interruptions or overlappings are slight enough not to disturb.

In contrast, uniform reproductions on a polyhedron of a low number of surfaces n are also possible; the uninterrupted coherence of the whole world map can be exactly determined mathematically by means of the reproduction rule. We have included two examples of this sort. SNo. 236 is the *perspective of the globe from its center point onto a hexadedron which is contiguous with it at six points.* The points are

($\phi = 0°$, $\lambda = 10°$W and 170°E)

($\lambda = 80°$E, $\phi = 60°$N and 30°S)

($\lambda = 100°$W, $\phi = 30°$N and 60°S)

The Coast and Geodetic Survey in Washington[106] has used this possibility in such a manner as to reproduce North America

undistorted on the map, which, however, is done at the cost of other parts of the earth. Africa is divided into three, Antarctica into two pieces widely divided from each other. The southern parts of both Indian peninsulae are far separated from Asia, as are the Pyrenean peninsula and half of Franch from Europe; and South America is rectangularly slit open down to its center.

The world map SNo. 237 (Illustration 24, plate VII) has been projected according to almost the same idea of reproduction, but it avoids all distortion of the lands of the earth. It is the *perspective of the globe from its center point onto the walls of an eight-cornered rectangular box* with quadratic lid and floor, which is contiguous with the globe at the two poles of the earth, while the four side walls are parallel to the meridian planes $\lambda = 20^{o}$ E and $\lambda = 70^{o}$ W at a distance from them equal to one and a half times the global radius.

The four meridian slits take their courses along the meridians 65^{o} and 155^{o} E, 25^{o} and 115^{o} W from the corners of the square surrounding the north pole from $25^{o}15'$ toward the south. They are to be drawn in two squares between 65^{o} and 155^{o}E and between 155^{o}E and 115^{o}W to $25^{o}15'$S. In the two other

squares, one draws them somewhat further toward the south, in order to attach the southern ends of Africa and South America uninterrupted in an unaltered projection. Every great circle follows a straight line course on each of the six leaves of the two projections No. 236 and 237, since the reproduction rule for the center-perspective projection (SNo. 1) applies for each of the leaves. The circles of latitude in the single leaves of projection No. 237 are circles on the lid and floor of the box, hyperbolae on the side walls; in projection No. 236 they are arcs of ellipses, parabolae and hyperbolae. The equations for lid and floor in No. 237 are as for the center-perspective earth-axical projection, $r = \mathrm{tg}\ \delta$; $\alpha = \lambda$, and on the side walls (λ calculated from the central meridian of the region), $y = 1.5\ \mathrm{tg}\ \lambda$; $x = 1.5\ \sec\lambda\ \mathrm{tg}\ \phi$.

SECTION III: A DIAGRAM OF PROPERTIES FOR MAP PROJECTION

In our system table, each projection proves to be a certain combination of characteristic properties. The table makes it possible to judge just which properties may be combined. On the other hand, it does not discern which

combinations of properties are impossible. This diagram gives information about both questions, about possibilities of combinations, as well as about the mutual exclusion of properties. Family V is not included in this examination, since the grids of that family exhibit various combinations of properties in particular parts of the map in a manner which is not uniform.

We examine the combination possibilities of the following fourteen properties:

Fourteen Characterizing Properties of Maps

Abbreviation and Designation	Definition
1. M = centrally-circular	The N are same-centered circles. To this group belong the whole of families I and II, nothing from family III, the cylinder-circular projections of family IV.
2. D = doubly-symmetrical	The grid is symmetrical to the basic line and to the central H. To this group belong the whole of families II and III, nothing from families I and IV.

Abbreviation and Designation	Definition
3. N = secondary circle-spaced	All N are equal-spaced. To these belong: Family I, No. 1 - 73; Family II, No. 83 - 152; and single projections from family III and IV.
4. Z = zenithal (cellular)	Z combines with M and N the property that all H are congruent with their divisions. To these belong: Family I, No. 1 - 58; Family II, No. 83 - 97; the cylindrical projections of family IV; nothing from family III. Z is a subdivision of M and N.
5. K = generally conical	Subdivision of Z with straight H. To these belong the projections named under Z, except No. 53 - 58.
6. R = radial (azimuthal)	Subdivision of K. All N are full circles. To these belong family I, No. 1 - 30, and none from other families.

Abbreviation and Designation	Definition
7. H = prime-point map	Subdivision of M. A prime-point is represented as center of the N. To these belong numerous projections from Family I, No. 92 - 99; No. 157 from Family II; No. 197, 198 from Family IV; Nothing from Family III.
8. A = interval-true	The N are either equal-intersecting circles or congruent, parallel displaced curves. Their intervals are equal to the arc intervals of their originals on the uniform globe. A is found in all the families except for Family III.
9. F = area-true	Each piece of surface of the map is area-equal to its original on the uniform-globe. F is found in all families.
10. W = shape-true	All curves of the map intersect at the same angles as on their originals on the uniform globe. W is found in all families.

Abbreviation and Designation	Definition
11. O = orthodromic (straight-directional)	All straight lines of the map are pictures of great circles of the globe. O is not possible in Family II.
12. L = loxodromic (course-linear)	All straight lines of the map are pictures of course-equals (loxodromes) of the globe. L is not possible in Families I and III.
13. G = counter-azimuthal	The straight lines through a shape-true map-point are pictures of the azimuth-equals which have the original of this point as direction-point. G is possible in all families.
14. P = perspective	The map is the perspective picture of the uniform globe on a cone which has the same axis and which can also deteriorate into a plane or cylinder, from a sighting-point on the common axis. P is not possible in Family III, but is possible in all other Families.

These fourteen properties can yet be ordered in two basically different groups. The first eight, M, D, N, Z, K, R, H, A, apply for a particular grid of H and N on the map; they are *grid properties*. On the other hand, the properties F, W, O, L, G and P apply for the map no matter which grid of H and N we want to consider for its construction; for, the areas, angles, and straight lines of the map, as well as its perspective orientation, do not change, even if we change over to another grid of H and N. These properties, 9 - 14, are *independent of the grid*. Thus, when one wants to combine several stipulations of the 1 - 8 group in a projection, one must do so with respect to the same grid. On the other hand, when fulfilling stipulations of the group 9 - 14 and at the same time from group 1 - 8, the choice of grid remains completely free. One must only state which grid is supposed to apply for the map as a means of designating such combination of properties. If, for example, we think of Hammer's center-interval-true counter-azimuthal projection (SNo. 78), then we naturally think of the polar grid with respect to its counter-azimuthality (G) since the concept of the counter-azimuth is connected in one's imagination with the meridians. But the polar grid in this map is not centrally circular (M) or interval-true (A); it is, rather, the grid of the shape-true

direction-point of azimuth-equals reproduced as straight lines. Or, if the grid of a perspective (P) projection is supposed to be neither centrally-circular (M) nor doubly-symmetrical (D), then one is obviously not talking about its prime-point, which is, of course, always supposed to be centrally-circular; but, no doubt, the lack of center-circularity and double-symmetry applies for the polar grid of an oblique-axical perspective projection.

The Property Table

Everything else will become evident in the following property table, which states all 84 really possible combinations from among the 14 properties and enumerates for each such combination which projections from our system table are thusly characterized. Each possible combination is shown in this table by crosses (+) underneath that property fulfilled in that projection. The lexicographic arrangement according to *field numbers* (FNo.) 1 - 84 is carried out in such a manner that that combination comes first which has crosses in the column more to the left.

Table to Diagram of Properties of Map Projections

[See pages 182a and 182b following.]

Parts of the System Table	Field No. in Diagram	Properties of Grids	Properties Independent of Grids
[Last column]	Examples from the System		

	Examples:
Family II Branch A, Subbranch A	No. 94,95 Etzlaub-Mercator No. 94 Prime-point grid of transverse-axical Mercator projection No. 96 Mercator map, projected perpendicular to map plane on plane through equator No. 93 Wetch, cylinder-perspective projection No. 97 Special cylindrical projection, No. 92 No. 90, 91 flat maps No. 87-89 Lambert and Behrman No. 83-86 Circle-perspective cylindrical projections
Family II Branch A, Subbranch B	No. 98 special cylinder-circular projection No. 99 Special cylinder-circular projection No. 100 Mercator (Sauson) No. 101,105,108,111,114,117,120,123,125, 128,130 after Apianus II, Winkel, Eckert, Donis, Wagner No. 102, 131-149 after Eckert, Collignon, Mollweide, Nell, Hammer, Wagner, Craster, Prepetit-Foulaut, Bourdin

No. 103-4, 106-7, 109-10, 115-16, 118-19, 121,123-24,126-27,129,150-152

Family II, Branch B, Subbranch IV, No. 215,216

No. 157 special cylinder-cirular projection

No. 215 in the case $\phi_0 = 0$, polar grid of special counter-azimuthal projection

No. 153, 154 Apianus I

No. 216 in the case $\phi_0 = 0$, polar grid of special counter-azimuthal projection

No. 3 Polar grid of the transverse-axical distant-perspective projection

No. 155, 156 Glareanus, van der Grinten, No. III

Family I, Branch A Subbranch A Order a, Class I

No. 24 Mercator (Postel)

No. 23 Lambert

No. 2 All-circular, Prime-point grid (Gnomonic)

No. 1 Center-perspective, prime-point grid (gnomonic)

No. 3-21 Outer-perspective, prime-point grid (external)

No. 22, 25-27 Radial introening after Lidman, Breusing, Airy, James and Clark

No. 30 radial ring map Maurer

No. 28,29 Radial ring maps Hammer, Maurer

No. 31 Special ring map

Family I, Branch A Subbranch A, Order a, Class II

No. 41-43 after Schjerning

No. 44 Lambert

No. 39,40 Lambert-Gaub

No. 32-27, 197, 198 Murdoch, Braun

No. 38 Special case of zone-perspective projections

No. 47-49 Ptolemaus, De L'Isle, Murdoch I

No. 51-52 Albers

No. 45-46 Zone-perspective, Murdoch III

Family I, Branch A
Subbranch A, Order b

No. 54, Special Projection, All H congruent arcs

No. 53, special projection, all H congruent arcs, Wiechel

No. 224, Projected on plane perpendicular to map plane, not parallel to equator or to central-meridian

No. 55 special projection all H congruent arcs

No. 56 special projection all H congruent arcs

No. 57 special projection all H congruent arcs

No. 58 special projection all H congruent arcs

Family I, Branch A,
Subbranch B, Family IV,
No. 207-214

No. 63 after Stab

No. 61, 67-69 Schjerning No. V, III, II

No. 65,72,73 Nell, Maurer

No. 66, special case of Schjerning No. V

No. 59 Mercator (Bonne)

No. 60 and 208, 210, 211

No. 61, 70, 71, 212, 213 Nell, Eckert, Collignon

No. 267 Mercator Maps Projected onto a plane through the equator from a sighting point in the finite

No. 62, 209, 214

Family I, Branch B
Family IV, No. 215,216

No. 78 Hammer, Grid of a special point

No. 74, 77 special radial-circular, non-zenithal grids

No. 79, 80 grid of special point

No. 75 special radial-circular, non-zenithal grid

No. 215, when $\phi_0 >< 0$

No. 81, 82 simplified conical projection

No. 216, when $\phi_0 >< 0$

No. 76 special radial-circular, non-zenithal grid

Family III

No. 176, 178 ordinary poly-conic and secondary-circular all-globular projection

No. 219, when $\phi_0 = 0$

No. 179, 183, 184, 186 area-true poly-conic and affined or Aitoff-ized area-true projection

No. 189 Littrow-Maurer

No. 170 polar or transverse-axical all-circular projection

No. 169, 171-173, 181, 190-194 shape-true circle-grids and projections by Fiorini, August, Eisenlohr, Guyou

No. 1 Polar grid of a transverse-axical center-perspective projection, affined

No. 196 Doubly-counter-azimuthal; No. 200, when $\phi_0 = 0$, Craig; No. 78-80 polar grids, when $\phi_0 = 0$

No. 3-21 polar grid of transverse axical outer-perspective projections

No. 158-168, 174, 175, 177, 180, 185, 187, 188, 195 Circle-grids, rectangular poly-conic projection, Aitoff, prime-circular all-globular, Schmidt, No. 199 for $\phi_0 = 0$

Family IV

No. 205 special projection, N parallel arcs

No. 206 special projection, N parallel arcs

No. 201, 202 special projection, N parallel arcs

No. 219 when $\phi_0 \gtrless 0$

No. 203,204 special projections, N parallel arcs

No. 2 polar grid of oblique-axical all-circular projections

No. 217, 218, 220-223 shape-true circle-grids, Peiree, Mercator, reproduced by reciprocal rays

No. 1 polar-grid of oblique-axical, center-perspective projections

- same as 80, affined

No. 225 projected from sighting-point within finite onto plane which is neither parallel to equator not to center-meridian

No. 78-80, 200, when $\phi_0 \gtrless 0$

No. 3-21 polar grid of transverse-axical projections

No. 199 Schoys radial-directional grid ($\phi_0 \gtrless 0$)

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TEXT NOT REPRODUCIBLE

Tabelle zum Begriffsschaubild der Kartenentwürfe

Teile der Systemtabelle	Feld-Nr. im Schaubild	Eigenschaften der Netze: M	D	n	z	K	r	R	a	Netzfreie Eigenschaften: f	w	o	l	g	p	Beispiele aus dem System
Stamm II, Ast α, Zweig A	1	+	+	+	+	+		+			+		+			Nr. 94, 95 Ebrlaub-Merkator
	2	+	+	+	+	+		+			+					Nr. 91 Hauptpunktnetz querachsigen Merkatorentwurfs
	3	+	+	+	+	+		+					+			Nr. 96 Merkatorkarte, senkrecht zur Kartenebene projiziert auf Ebene durch den Äquator
	4	+	+	+	+	+		+							+	Nr. 93 Wetch, säulenansichtiger Entwurf
	5	+	+	+	+	+		+								Nr. 97 Besonderer säuliger Entwurf, Nr. 92
	6	+	+	+	+	+			+							Nr. 90, 91 Plattkarten
	7	+	+	+	+	+				+						Nr. 87--89 Lambert und Behrmann
	8	+	+	+	+	+										Nr. 83--86 Kreissichtige säulige Entwürfe
Stamm II, Ast α, Zweig B	9	+	+	+				+		+						Nr. 98 Besonderer säulenkreisiger Entwurf
	10	+	+	+				+								Nr. 99 " " "
	11	+	+	+					+	+						Nr. 100 Merkator (Sanson)
	12	+	+	+					+							Nr. 101, 105, 109, 111, 114, 117, 120, 123, 125, 128, 120 nach Aplanus II, Winkel, Eckert, Denis, Wagner
	13	+	+	+						+						Nr. 102, 131--149 nach Eckert, Collignon, Mollweide, Nell, Hammer, Wagner, Craster, Prépetit-Foucaut, Bourdin
	14	+	+	+												Nr. 103/04, 106/07, 109/10, 112/13, 115/16, 118/19, 121, 123/24, 126/27, 139, 150--152
Stamm II, Ast β, Stamm IV, Nr. 215, 216	15	+	+					+								Nr. 157 Besonderer säulenkreisiger Entwurf
	16	+	+						+					+		Nr. 215 im Falle $\varphi_0 = 0$. Polnetz besonderen gegenabstandstreuen Entwurfs
	17	+	+						+							Nr. 153, 154 Aplanus I
	18	+	+											+		Nr. 216 im Falle $\varphi_0 = 0$. Polnetz besonderen gegenabstandstreuen Entwurfs
	19	+	+												+	Nr. 8 Polnetz des querachsigen fernsichtigen Entwurfs
	20	+	+													Nr. 155, 156 Clarenus, van der Grinten Nr. III
Stamm I, Ast α, Zweig A, Ordnung a, Klasse I	21	+		+	+	+	+	+	+							Nr. 24 Merkator (Postel)
	22	+		+	+	+	+	+		+						Nr. 23 Lambert
	23	+		+	+	+	+	+			+				+	Nr. 2 Allkreisig, Hauptpunktnetz (Stereographisch)
	24	+		+	+	+	+	+				+			+	Nr. 1 Mittensichtig, Hauptpunktnetz (Gnomonisch)
	25	+		+	+	+	+	+							+	Nr. 3--21 Außensichtig, Hauptpunktnetz (Extern)
	26	+		+	+	+	+	+								Nr. 22, 25--27 radlich vermittelnd nach Lidmen, Breusing, Airy, James und Clarke
	27	+		+	+	+	+		+							Nr. 20 radliche Ringkarte Maurer
	28	+		+	+	+	+			+						Nr. 28, 29 radliche Ringkarten Hammer, Maurer
	29	+		+	+	+	+									Nr. 31 Besondere Ringkarte
Stamm I, Ast α, Zweig A, Ordnung a, Klasse II	30	+		+	+	+		+	+							Nr. 41--43 nach Schjerning
	31	+		+	+	+		+		+						Nr. 44 Lambert
	32	+		+	+	+		+			+					Nr. 32, 40 Lambert-Gauß
	33	+		+	+	+		+								Nr. 32--37, 197, 198 Murdoch, Braun
	34	+		+	+	+		+					+			Nr. 38 Sonderfall zonensichtiger Entwürfe
	35	+		+	+	+			+							Nr. 47--49 Ptolemäus, De l'Isle, Murdoch I
	36	+		+	+	+				+						Nr. 51--52 Albers
	37	+		+	+	+										Nr. 45/46 zonensichtig, 50, Murdoch III
Stamm I, Ast α, Zweig A, Ordnung b	38	+		+	+			+	+							Nr. 52 Besonderer Entwurf. Alle R kongruente Kreisbogen
	39	+		+	+			+		+						Nr. 53 Wiechel " " " "
	40	+		+	+			+					+			Nr. 224 Merkatorkarte, senkrecht zur Kartenebene projiziert auf Ebene, die weder zum Äquator noch zum Mittelmeridian parallel ist
	41	+		+	+			+								Nr. 55 Besonderer Entwurf. Alle R kongruente Kreisbogen
	42	+		+	+				+							Nr. 56 " " " " " "
	43	+		+	+					+						Nr. 57 " " " " " "
	44	+		+	+											Nr. 58 " " " " " "

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TEXT NOT REPRODUCIBLE

Tabelle zum Begriffsschaubild der Kartenentwürfe (Fortsetzung)

Teile der Systemtabelle	Feld-Nr. im Schaubild	Eigenschaften der Netze								Netzfreie Eigenschaften						Beispiele aus dem System
		M	h	a	z	K	r	H	s	f	w	o	l	g	p	
Stamm I, Ast a, Zweig B; Stamm IV, Nr. 207–214	45	+		+				+	+	+						Nr. 63 nach Stab
	46	+		+				+	+							Nr. 64, 67–69 Schjerning Nr. V, III, II
	47	+		+				+		+						Nr. 65, 72, 73 Nell, Maurer
	48	+		+				+								Nr. 66 Sonderfall von Schjerning Nr. V
	49	+		+					+	+						Nr. 59 Merkator (Bonne)
	50	+		+					+							Nr. 60 u. 208, 210, 211
	51	+		+						+						Nr. 61, 70, 71, 212, 213 Nell, Eckert, Collignon
	52	+		+									+			Nr. 207 Merkatorkarten, projiziert auf Ebene durch den Äquator aus Augpunkt im Endlichen
	53	+		+												Nr. 62, 209, 214
Stamm I, Ast b; Stamm IV, Nr. 215, 216	54	+						+	+					+		Nr. 78 Hammer, Netz des Sonderpunkts
	55	+						+	+							Nr. 74, 77 Besondere radkreisige, nicht zenitale Netze
	56	+						+						+		Nr. 79, 80 Netz des Sonderpunkts
	57	+						+								Nr. 75 Besonderes radkreisiges, nicht zenitales Netz
	58	+							+					+		Nr. 215, wenn $\varphi_0 \gtrless 0$
	59	+							+							Nr. 81, 83 Vereinfachter Kegelentwurf
	60	+												+		Nr. 216, wenn $\varphi_0 \gtrless 0$
	61	+														Nr. 76 Besonderes radkreisiges, nicht zenitales Netz
Stamm III	62		+	+												Nr. 176, 178 Gewöhnlicher polykonischer und nebenkreisig allglobularer Entwurf
	63		+							+				+		Nr. 219, wenn $\varphi_0 = 0$
	64		+							+						Nr. 179, 183, 184, 186 Flächentreuer polykonischer und affinierter oder altoffierter flächentreuer Entwurf
	65		+								+			+		Nr. 189 Littrow-Maurer
	66		+								+				+	Nr. 170 Polnetz queraxiger allkreisigen Entwurfs
	67		+								+					Nr. 169, 171–173, 181, 190–194 Winkeltreue Kreisnetze und Entwürfe von Fiorini, August, Eisenlohr, Guyou
	68		+									+			+	Nr. 1 Polnetz queraxigen, mittelsichtigen Entwurfs
	69		+									+				Nr. 152 Polnetz queraxigen, mittelsichtigen Entwurfs, affiniert
	70		+											+		Nr. 185 Doppelt-gegenazimutal; Nr. 206, wenn $\varphi_0 = 0$, Craig; Nr. 78–80 Polnetze, wenn $\varphi_0 = 0$
	71		+												+	Nr. 3–21 Polnetz queraxiger außensichtiger Entwürfe
	72		+													Nr. 159–163, 174, 175, 177, 180, 195, 187, 193, 195 Kreisnetze, rechtschnittiger polykonischer Entwurf, Altoff, hauptkreisig-allglobular, Schmidt, Nr. 199 für $\varphi_0 = 0$
Stamm IV	73			+					+							Nr. 205 Besonderer Entwurf. Die N parallele Kreisbogen
	74			+												Nr. 202 " " " " " "
	75								+							Nr. 201, 203 " " " " " "
	76									+				+		Nr. 219, wenn $\varphi_0 \gtrless 0$
	77									+						Nr. 203, 204 Besondere Entwürfe; die N parallele Kreisbogen
	78										+				+	Nr. 2 Polnetz schiefachsiger allkreisiger Entwürfe
	79										+					Nr. 217, 219, 220–223 winkeltreue Kreisnetze; Peirce; Merkator, abgebildet durch Kehrwertstrahlen
	80											+			+	Nr. 1 Polnetz schiefachsiger mittelsichtiger Entwürfe
	81											+				— " " " " , affiniert
	82												+			Nr. 225 Merkator aus Augpunkt im Endlichen projiziert auf Ebene, die weder zum Äquator noch zum Mittelmeridian parallel ist
	83													+		Nr. 78–80, 200, wenn $\varphi_0 \gtrless 0$
	84														+	Nr. 3–21 Polnetz schiefachsiger außensichtiger Entwürfe
	85															Nr. 199 Schoys radstrahliges Netz ($\varphi_0 \gtrless 0$)

Clarification of the Diagram of Properties

A graphic presentation of the contents of this table is offered in the Diagram of Properties, Illustration 25 (plate X). It shows 15 areas, designated by means of the varying markings of its border line and the identifying letters M,D,N,Z,K,R, H,A,F,W,O,L,G, and P. (There are two areas for O.) The areas for a top have a tube-shape and fine border lines; the others, M to H, have curved, often indented, border lines with various kinds of signature. The 15 border lines enclose or exclude each other or touch or intersect each other in many various ways, so that 84 fields with identifying *field numbers* arise, each of which is enclosed by at least one of the 15 properties. Each field represents the combination of those properties inside of whose identifying lines it occurs; thus the field corresponds to that line in the property table on which those properties are marked with a cross.

For example, field 39 lies inside the border lines M,N,Z, H, and F, but outside all others. Thus it corresponds to the zenithal area-true prime-point-maps, which, however, are not generally-conical. As an example for this field under FNo. 39, the table gives the Wiechel Projection, SNo. 53. Field 49 lies inside the border lines M,N,A,F but outside all others. The corresponding projections are, then, secondary-circle-spaced,

centrally-circular, area-true and interval-true, but they are neither prime-point-maps nor zenithal. For this field, the property table gives under FNo. 49 the example of the Mercator-Bonne Projection, SNo. 59. Field 72 refers to projections which are doubly-symmetrical (D) but otherwise exhibits none of the other 13 properties, as do, for example, the globular projection, SNo. 158, and many others.

The diagram allows one to get a comprehensive view of a great number of facts about cartography with one glance. One sees immediately that true-shape cannot be combined with either true-area or equidistance, although with every other of our 14 map properties; that loxodromic maps can be neither radial nor perspective, neither straight-directional nor counter-azimuthal, but can exhibit all 9 other properties. Counter-azimuthality can be combined with equidistance, true-area and true-shape; counter azimuthal maps can also be centrally-circular, double-symmetrical and prime-point-maps, but neither secondary-circle-spaced nor zenithal.

That many special combination possibilities of the last four properties, O, L, G, P, with others are based on the following: In perspective projections (P), the *polar grid* can exhibit, according to whether its axical inclination is earth, transverse or oblique, very different forms of sharply

changing regularity (FNo. 4, 19, 23, 24, 25), even to the point of lacking any other property (FNo. 84); the doubly-symmetrical, cylinder-circular ring map polar grid of the transverse-axical, distant-perspective projection (FNo. 19) is especially noteworthy. In three instances (all-circular projections), the perspective polar grids are shape-true (FNo. 23, 66, 78), and in three others they are (central-perspective projection) straight-directional (FNo. 24, 69, 81). One does not obtain perspective projections, but rather straight-directional projections (FNo. 84) by means of colinear transformation of the polar grid of a transverse-axical, center-perspective projection. The variety of loxodromic projections results from colinear reproductions of Mercator projections; and it has already been pointed out on p.   that very different sorts of properties can be affixed to counter azimuthality, which is actually only an equation for the coordinates of a polar grid.

Making excursions from field to field in the diagram is very stimulating and educational! Neighboring fields, which are divided by only one simple border line, are different by only one property, so that one can begin at one sort of projection and wander through gradually changing levels, arriving, then, at projections of completely properties. If, for example,

one goes from field 1 (e.g., the Etzlaub-Mercator-Map), in which 8 stipulations (M,D,N,Z,K,H,W,L) are fulfilled, and moves always toward the left, then he finds that in field 2 (e.g., Lambert Gauss = transverse-axical Mercator projection) only the course-linearity (l) is lost, since the loxodromes in that field are not straight lines. Then the shape-true conical projection follow in field 32 with the loss of double-symmetry (D), whereupon in field 23 the now all-circular projection becomes both radial (r) and perspective (p) and still maintains true-shape (w). This true-shape is then lost in the neighboring field 25 (an outer-perspective projection); but in field 24 (a centrally-perspective projection) one can trade in straight-directionality (O) for true-shape. Now follows field 26, still radial (r), but no longer perspective (P). Equidistance (A) joins up in field 21 (Mercator-Postel Projection), which in field 22 and 23 is substituted for by true-area. Field 22 yields, as all fields up to now, prime-point-maps H (Lambert's projection SNo. 23), but field 28 is for ring-maps. Radial ring maps without special properties follow in field 29; in field 37 are yet conical, and in field 44 still more conical, although still zenithal, ring maps, whereas there is only center-circularity (M) in fields 53 and 61, combined in field 53 with being secondary-circle-spaced (N).

In a similar manner, a hike through the tube-shaped areas of the interval-true (a) or area-true (f) projections is interesting.

The field numbers are also included in the large system table in the second column. The relationship of the system with the diagram becomes especially clear in the following manner: The field area of the *Family* I (SNo. 1 - 82, FNo. 21 - 61) is marked with brown in the diagram, and the area of its branch A (subbranch A = SNo. 1 - 58, FNo. 21 - 44; subbranch B = SNo. 59 - 73, FNo. 45 - 53) with two kinds of brown oblique striations, the area of its branch B (SNo. 74 - 82, FNo. 54 - 61) with brown borders in two parts of the surface. The field area of *Family* II is designated by red color, in the area of its branch A (subbranch A = SNo. 83 - 97, FNo. 1 - 8; subbranch B = SNo. 98 - 152, FNo. 9 - 14) with two kinds of red oblique striation, and in the area of its branch B (SNo. 153 - 157, FNo. 15 - 20) with red border. *Family III* (SNo. 158 - 196, FNo. 62 - 72) is designated by means of the red-brown border of two parts of the surface of the diagram. Its branches are not singled out, since row-circularity has not been found useful among the 14 properties of our diagram. The outer-most border fields (FNo. 73 - 84) belonging to *Family IV*, and the FNo. 85 lying outside all border lines and not fulfilling any of

our 14 properties, are not designated by any color.

The last column of the table to the diagram contains all 225 SNo.'s of families I - IV of our system table. Each fulfills at least one of the 14 characterizing properties of the diagram with the exception of the only radial-directional grid of projection SNo. 199 by Schoy, which has been assigned to the marginal space FNo. 85 in the general case $\phi_0 \gtrless 0$

Index of Germanizations

(On the left are the German germanizations, from the Latin or Greek derived term to the "German" word. On the right is the English literal equivalent to this process. The translator has reordered Maurer's list so that the English is in alphabetical order.)

General

Zentrum = Mitte	center = center, middle
koaxial = gleichachsig	coaxial = equal-axical, of the same axis
konzentrisch = gleichmittig	concentric = equal-centered, same-centered
Konus = Kegel	cone = cone
Zylinder = Saule	cylinder = cylinder (or pillar)

aquidistant = gleichabstandig	equidistant = equal-interval, equidistant
homogen = gleichteilig	homogeneous = of equal parts, equal-spaced
orthogonal = rechtschnittig	orthogonal = rectangular
parallel = gleichlaufend	parallel = parallel
Projektion = Entwurf	projection = projection, reproduction, picture
Quincunx = Funfform	quincunx = quincunx, five-form
Radius = Halbmesser	radius = radius
Sektor = Kreisausschnitt	sector = sector, sector of a circle
Segment = Kreisabschnitt	segment = segment, segment of a circle
Symmetrie = Spiegelgleichheit	symmetry = symmetry
Tangentialzylinder = Beruhrsaule	tangential cylinder = tangential cylinder, contiguous cylinder
Transformation durch reziproke Radien = Umwandlung durch Kehrwertstrahlen	transformation by means of reciprocal radials = transformation by means of reciprocal rays

Designation of Lines

Aquideformaten = Verzerrungsgleichen	equiform = distortion-equal, of same distortion
Horizontalkreise = Nebenkreise	horizontal circles (parallels) = secondary-circles (abbr.: N)

Loxodromen = Kursgleichen	loxodrome = course-equal
Orthodromen = Gro kreise	orthodrome = great circle
Orthogonale Trajektorien = Rechtschnittgleichen	orthogonal trajectory = rectangular-intersection-equals
Parallelen = Breitenkreise	parallels = circles of latitude

Inclination of Projection

horizontal (zenithal) = schiefachsig	horizontal (zenithal) = oblique-axical
normal (Polarprojektion, Aquatorprojektion) = erdachsig	normal (polar projection, equator projection) = earth-axical
transversal (Aquatorialprojektion, Meridianprojektion = querachsig	transversal (equatorial projections, meridian-projections) = transverse-axical

Designation of Projections

achsial = mittkreisig	axial = axial, centrally-circular, center-circular
azimuthal = radlich (radkreisig und radstrahlig)	azimuthal = radial (radial-circular and radial-directional)
konform (autogonal, isogonal, orthomorph) = winkeltreu	conform (autogonal, isogonal, orthomorphic) = shape-true
konisch = kegelig (kegelkreisig und kegelstrahlig)	conical = conical (conical-circular, conical-radial

zylindrisch = saulig	cylindrical = cylindrical
doppelsymmetrisch = doppelspiegelig	double-symmetrical = double-symmetrical, doubly-symmetrical
aquidistant = abstandstreu (punktabstandstreu und kreisabstandstreu)	equidistant = interval-true, distance-true (point-interval-true and circle-interval-true)
aquivalent (isomer, authalisch, homalographisch) = flachentreu	equivalent (isomeric, auto-, homalographic) = area-true
extern = au ensichtig	external = external-perspective, outer-perspective
geometrisch einfach definiert = allgemein kegelig	geometric, simply defined = generally-conical
gnomonisch (Zentralsperspektive) = mittensichtig	gnomonic (central-perspective) = center-perspective, central-perspective
isozylindrisch = flachentreu saulig	isocylindrical = area-true cylindrical
isopharischstenoter = flachentreu kegelig	isopheric stenoter = area-true conical
loxodromisch = kurslinig	loxodromic = loxodromic, course-linear
orthodromisch = geradwegig	orthodromic = orthodromic, straight-directional
orthographisch = fernsichtig	orthographic = orthographic, distant-perspective

perspektivisch = sichtig (ebensichtig, kegelsichtig, saulensichtig)	perspective = perspective (plane-perspective, cone-perspective, cylinder-perspective)
polykonisch = allkegelig	poly-conic = all-conical
quinkunktial = funfformig	quincuncail = quincuncial, five-form
unecht konisch (konventionelle Kegelprojektion) = kegelkreisig	untrue conical (conventional conical projection) = conical-circular
unecht zylindrisch (konventionelle Zylinderprojektion) = saulenkreisig	untrue-cylindrical (conventional cylindrical projection) = cylindrical-circular, cylinder-circular
zenital = facherig	zenithal = zenithal, cellular

Index of Authors of Projections

Index of Illustrations

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(For translation of notes containing remarks, see pages 197 and 198 following.)

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[13]) Ha., S. 9, Anmerkung. „In echtzenitalen Projektionen (z. B. der von Wiechel) sind die Linien gleicher Verzerrung Kreise um den Kartenmittelpunkt (die Horizontalkreisbilder)."

[14]) Maurer, A. d. H. 1905, S. 355—67.

[15]) Z.-B., S. 59. „Aus der Eigenschaft der Zenitalität folgt, daß alle Punkte eines gleichen Zenitabstandes δ, die, wie auf der Kugel auch auf der Karte auf einem Kreise liegen, die gleichen Verzerrungen erleiden."

[16]) G., S. 234. „Eine Abbildung heißt zenital, wenn alle Punkte P, die auf der Erdkugel gleichweit von einem gewissen Zentralpunkte A abstehen, auch auf der Karte auf dem Umfang eines Kreises liegen, dessen Zentrum die Projektion A' jenes Punktes A ist, und wenn alle durch A gehende größte Kreise durch Gerade dargestellt werden, die sich in A' unter denselben Winkeln schneiden wie die größten Kreise in A."

[17]) [18]) Ha., S. 9. „Zu den nicht azimutalen Zenitalprojektionen -- die Parallelkreise durch konzentrische Kreise, die Meridiane aber nicht durch Radien der letzteren, sondern durch Kurven dargestellt — gehört der herzförmige Entwurf von Stab-Werner" (Nr. 63 unseres Systems). „Dieser Spezialfall könnte Veranlassung geben zur Trennung dieser nicht azimutalen Zenitalprojektionen in *unechte* (Umfang eines beliebigen Parallelkreises [allgemein Horizontalkreises] nicht durch den *ganzen* Umfang eines Kreises dargestellt, aber immer noch der Pol [allgemein Kartenmittelpunkt] Anfangspunkt der Streckenmessung auf den Meridianen), wozu der Stabsche Entwurf gehören würde, und in *echtzenitale*, die z. B. durch die Wiechelsche Projektion (Nr. 53 unseres Systems) repräsentiert wären."

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[76]) A. d. H. 1919, S. 22; 1920, S. 155.
[77]) Thorade, A. d. H. 1919, S. 35.
[78]) Maurer, A. d. H. 1919, S. 76; 1921, S. 116; Pet. M. 1920, S. 57. Z. f. V. 1922, S. 14.
[79]) Hammer, Pet. M. 1892, S. 65.
[80]) D. u. A., S. 150.
[81]) Nachtrag zum Lehrbuch der Navigation. Hrg. v. Reichswehrministerium, Berlin 1933.
[82]) T.-H., S. 196.
[83]) T.-H., S. 198ff., Nr. 147, 148.
[84]) Maurer, A. d. H. 1905, S. 125 u. 323.
[85]) Wedemeyer, A. d. H. 1919, S. 183, und Maurer, A. d. H. 1919, S. 278.
[86]) J. J. Littrow: Chorographie, Wien 1833, S. 142.
[87]) Wedemeyer, A. d. H. 1918, S. 209.
[88]) Fiorini: Le projezioni delle carte geografiche. Bologna 1881, S. 526.
[89]) Marine-Luftflotten-Rundschau, Berlin 1933.
[90]) F. August, Z. f. E. 1874, S. 1.
[91]) Eisenlohr, Crelle's Journal 1870, S. 143.
[92]) T.-H., S. 206.
[93]) D. u. A., S. 160, Fig. 74.
[94]) Peirce, American Journal of Mathematics, 1879, S. 394.
[95]) Guyou, Annales hydrographiques, 1887, S. 16.
[96]) Schoy, A. d. H. 1913, S. 33.
[97]) Maurer, Pet. M. 1914, II, S. 61.
[98]) Maurer, Pet. M. 1914, II, S. 117.
[99]) Craig, Map Projections, Kairo 1909.
[100]) Z.-B., S. 221, Fig. 150.
[101]) Maurer, A. d. H. 1919, S. 20.
[102]) Maurer, Pet. M. 1911, S. 91.
[103]) Pet. M., Erg.-Heft 16, S. 67.
[104]) Z.-B., S. 149.
[105]) D. u. A., S. 62.
[106]) D. u. A., S. 52, Fig. 46.

Translation of notes containing remarks.

[12] Zoppritz: Leitfaden der Kartenentwurfslehre, Leipzig 1884, S. 27. "...projections, which are called zenithal because changes take place at all points of equal zenith-distance from the center..."

[13] Ha., S. 9, note. "In true-zenithal projections (e.g., the one by Wiechel) the lines of equal distortion are circles about the center-point of the map (pictures of horizontal circles).

[15] Z.-B., S. 59. "It follows from the property of zenithality that all points of an equal zenithal-interval , which also lie on a circle of the map as on the globe, suffer the same distortion."

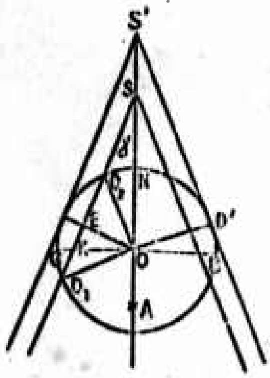
[16] G., S. 234. "A projection is called zenithal if all points P, which are at an equal distance from a particular central point A on the globe, lie also on the circumference of a circle on the map, the center of which circle is a projection A' of that point A--even when all greatest circles passing through A are represented by straight lines which intersect at the same angle at A' as do the greatest circles at A."

[17] [18]Ha., S. 9. "The heart-shaped projection by Stab-Werner belongs to the non-azimuthal zenithal projections--the parallel circles represented by concentric circles, the meridians, however, not by radials of the latter, by by curves" (No. 63 of our system). "This special case could give rise to the dividing of these non-

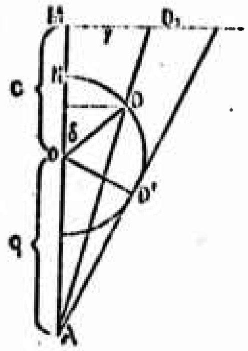
azimuthal zenithal projections into *untrue-zenithal* (circumference of an arbitrary parallel circle [a general horizontal circle] not represented by the *entire* circumference of a circle, but still the pole [general center point of the map] beginning point of the measurement of intervals on the meridians), to which group the Stab projection would belong, and *true-zenithal* projections, which, e.g., would be represented by the Wiechel projection" (No. 53 in our system).

[19] Hammer, Pet. M. 1915. "Thus, it would most likely be better, in place of the custom until now (I have also noted this custom), to reserve the name zenithal for such projections as reproduce the infinitely small region of the map center point congruently, as in Peirce's projection (No. 218 in our system) and in every azimuthal projection." (Strangely enough, this author then proceeds *not* to grant zenithality to the transverse-axical cylindrical projection which has just been recognized as reproducing the area around the pole congruently.)

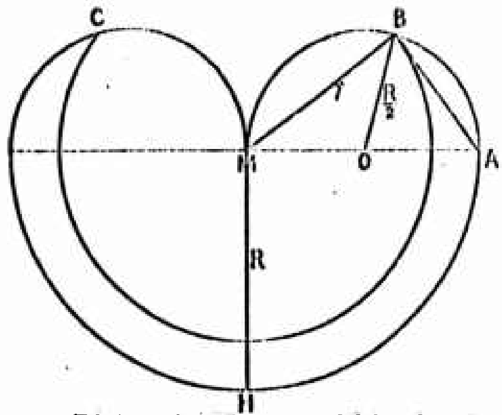
Plate I



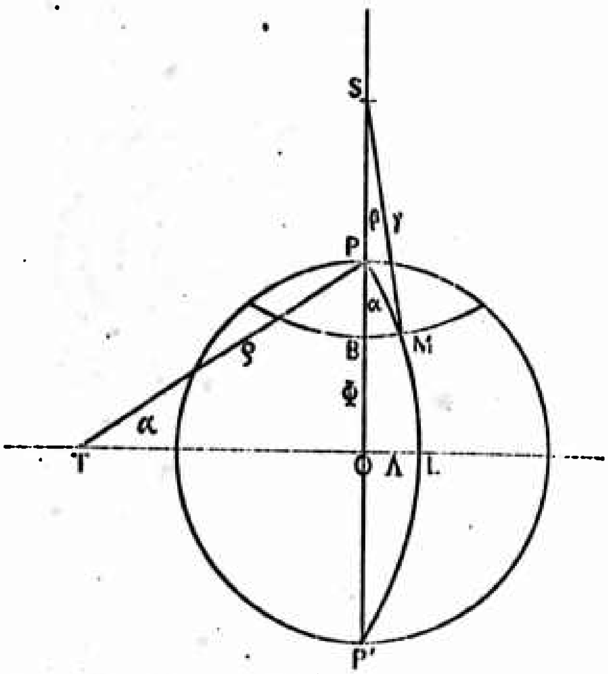
Ill. 1. Formation
of conical projections



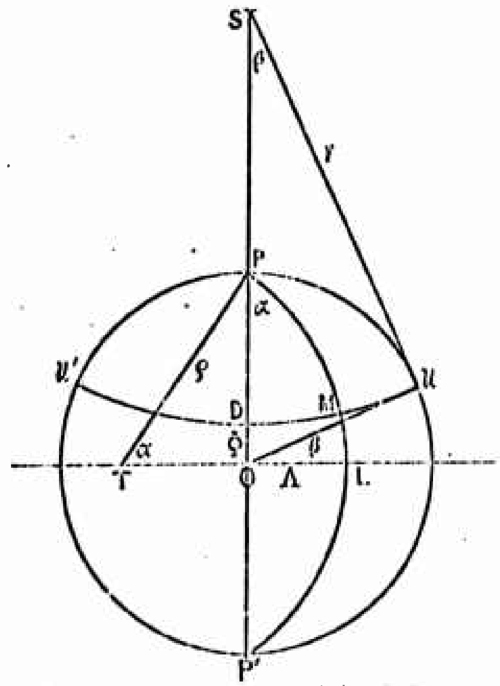
Ill. 2. Formation
of plane-perspective
projections



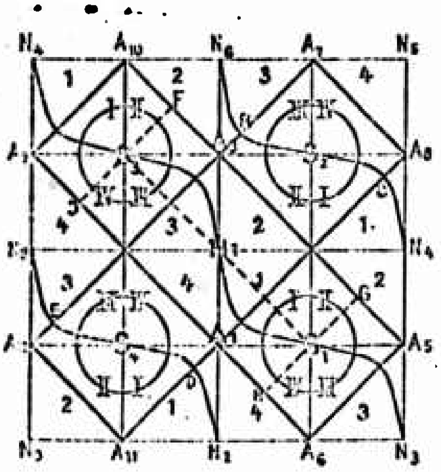
Ill. 3.
Re: projection No. 3



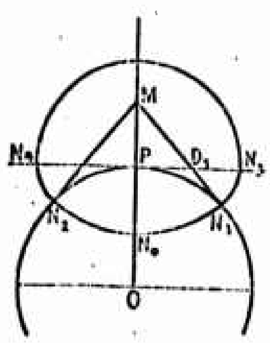
Ill. 5. Formation of
row-circular projections



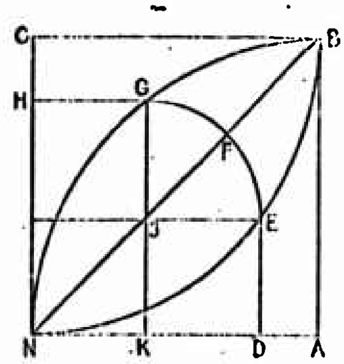
Ill. 7. Shape-true circle-
grid of class I



Ill. 14. Arrangement of the
Pierce quincuncial grid
(No. 218)

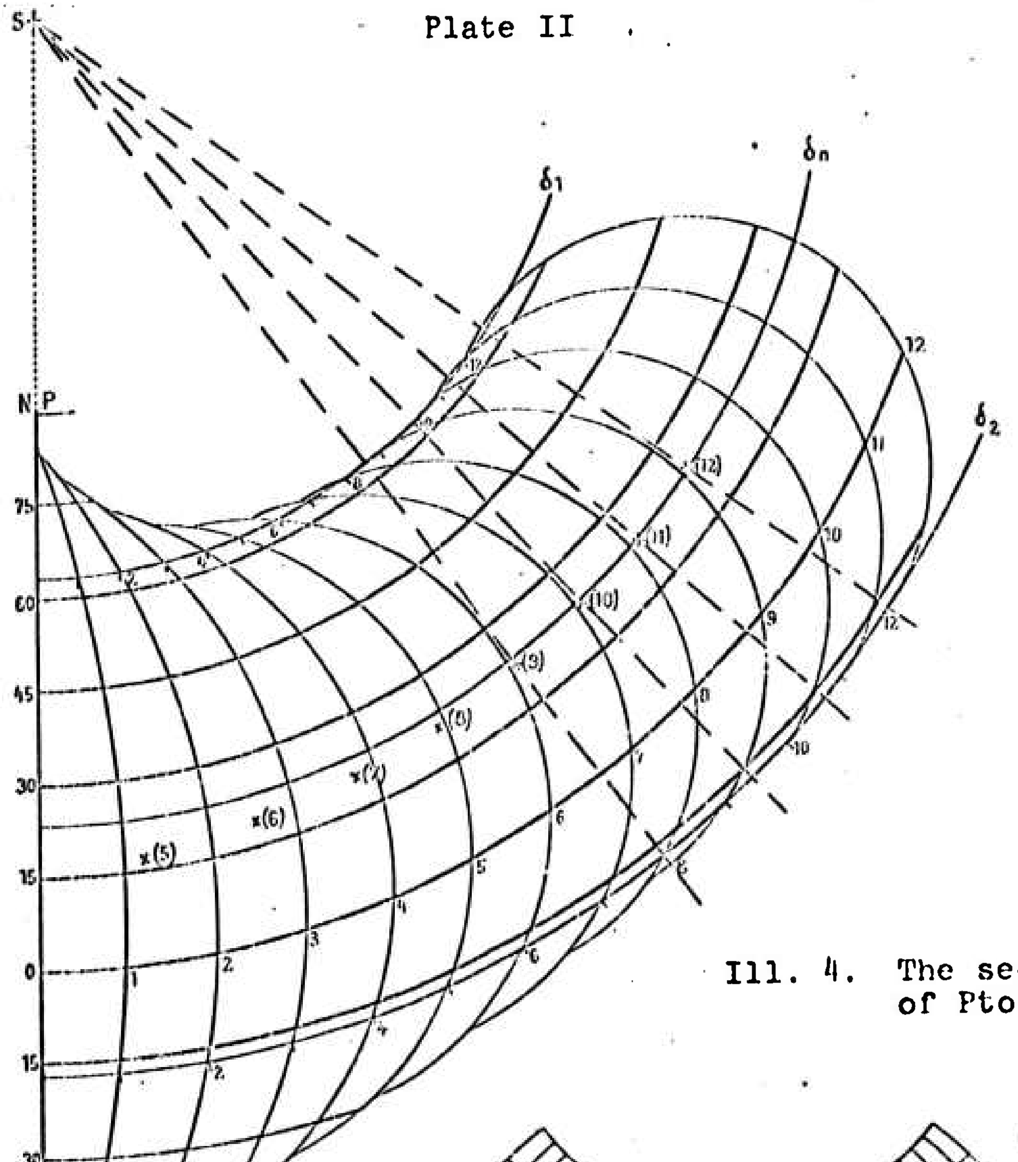


Ill. 17.
Re: projection
No. 227

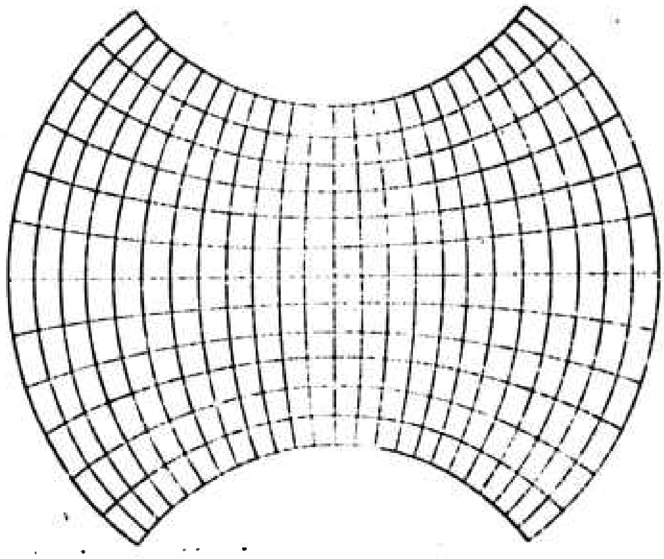


Ill. 22.
Re: projection
No. 234

Plate II

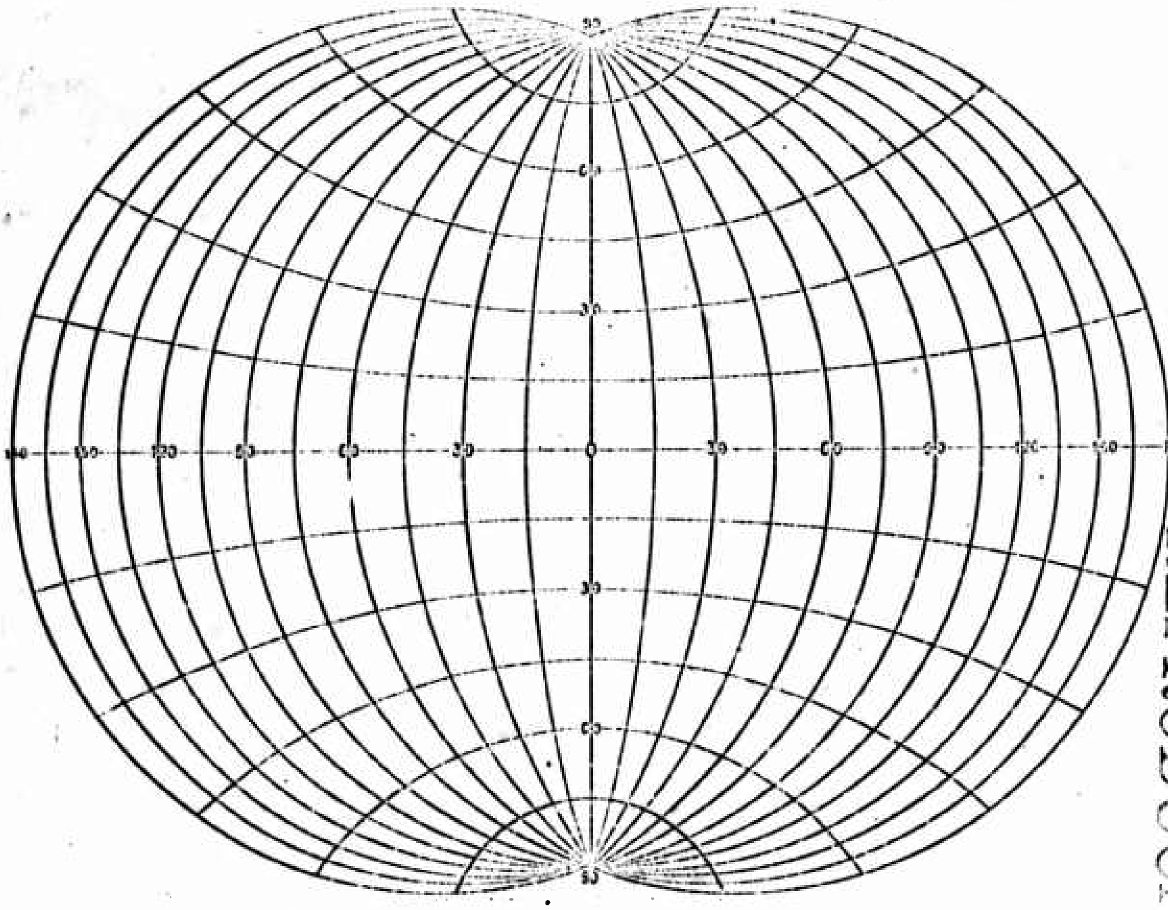


Ill. 4. The second projection of Ptolemaeus

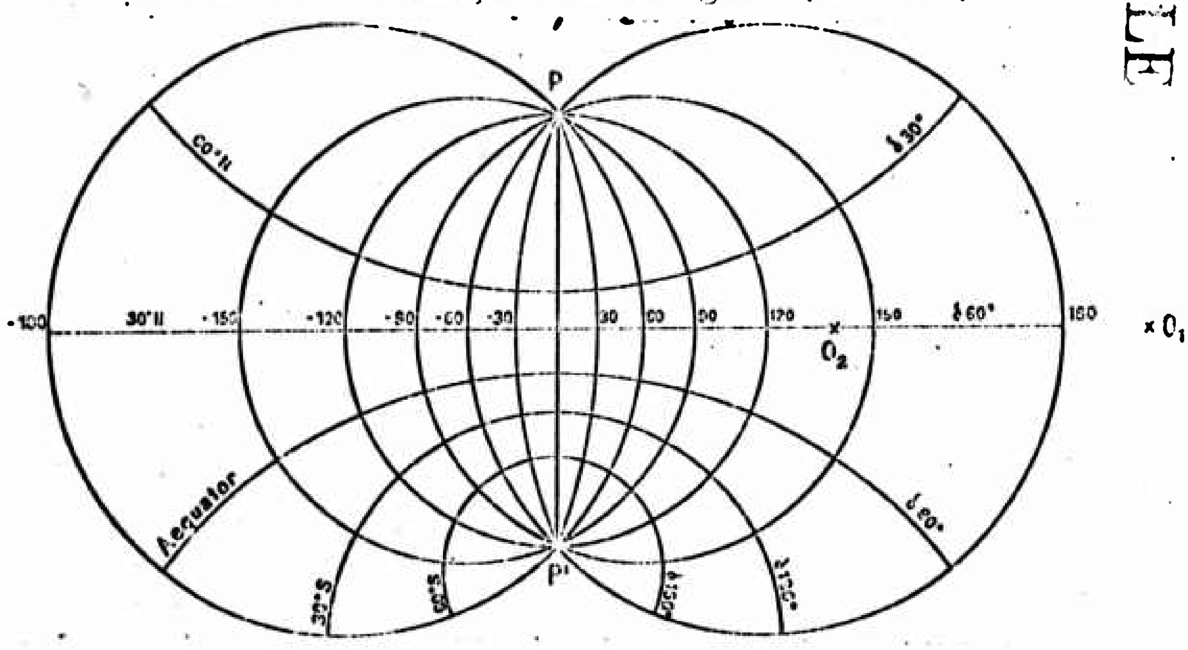


Ill. 11. Rectangular circle-grid with polar-lines (No. 180)

Plate III



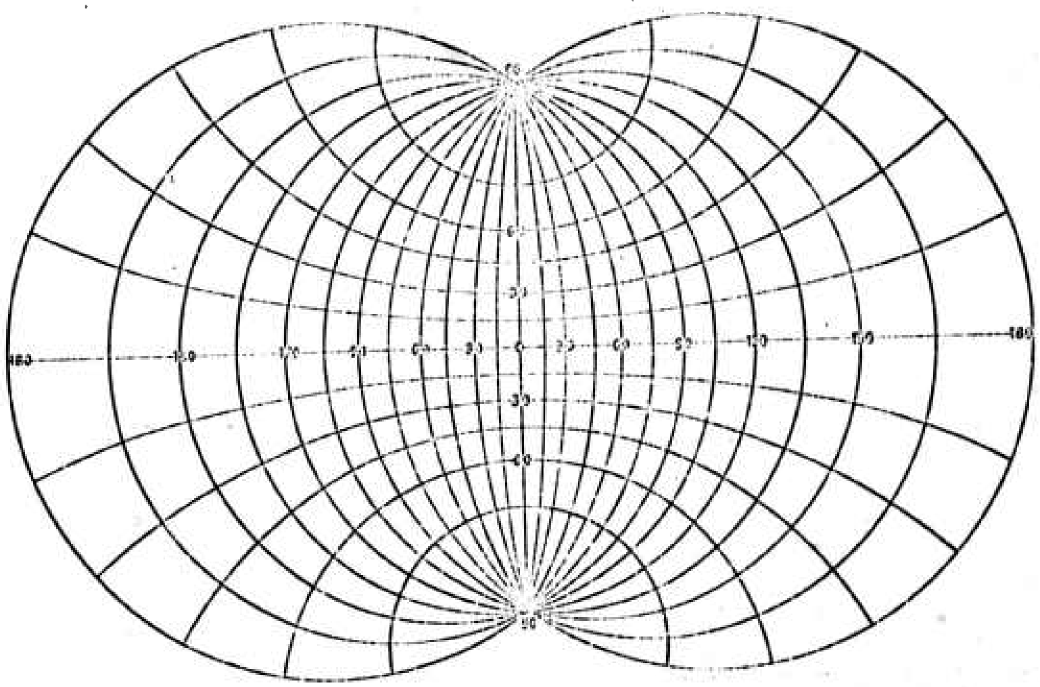
Ill. 6. Cell-equal circle-grid (No. 164)



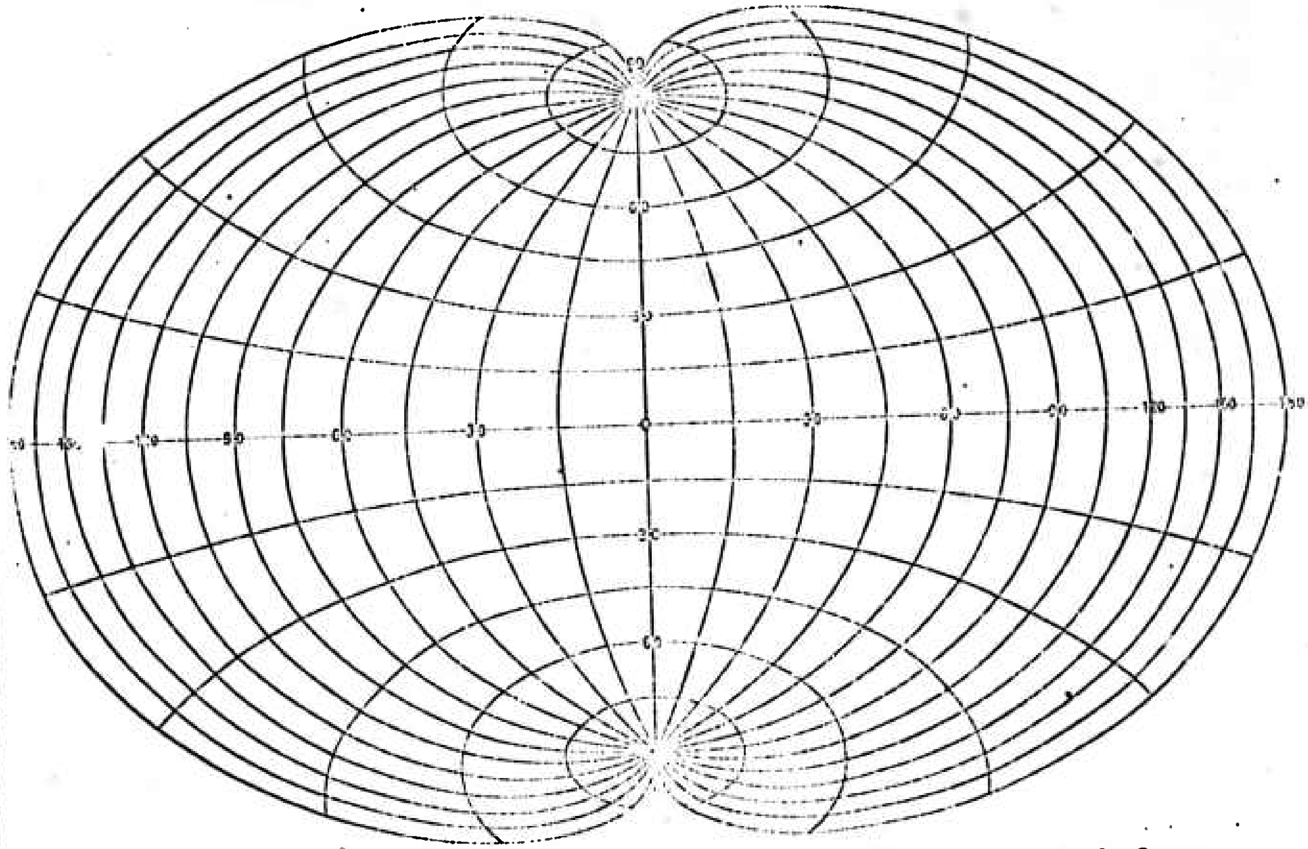
Ill. 8. Simple symmetrical shape-true circle-grid (No. 217)

(East-"Ost"
E - "O" pl.)

Plate IV

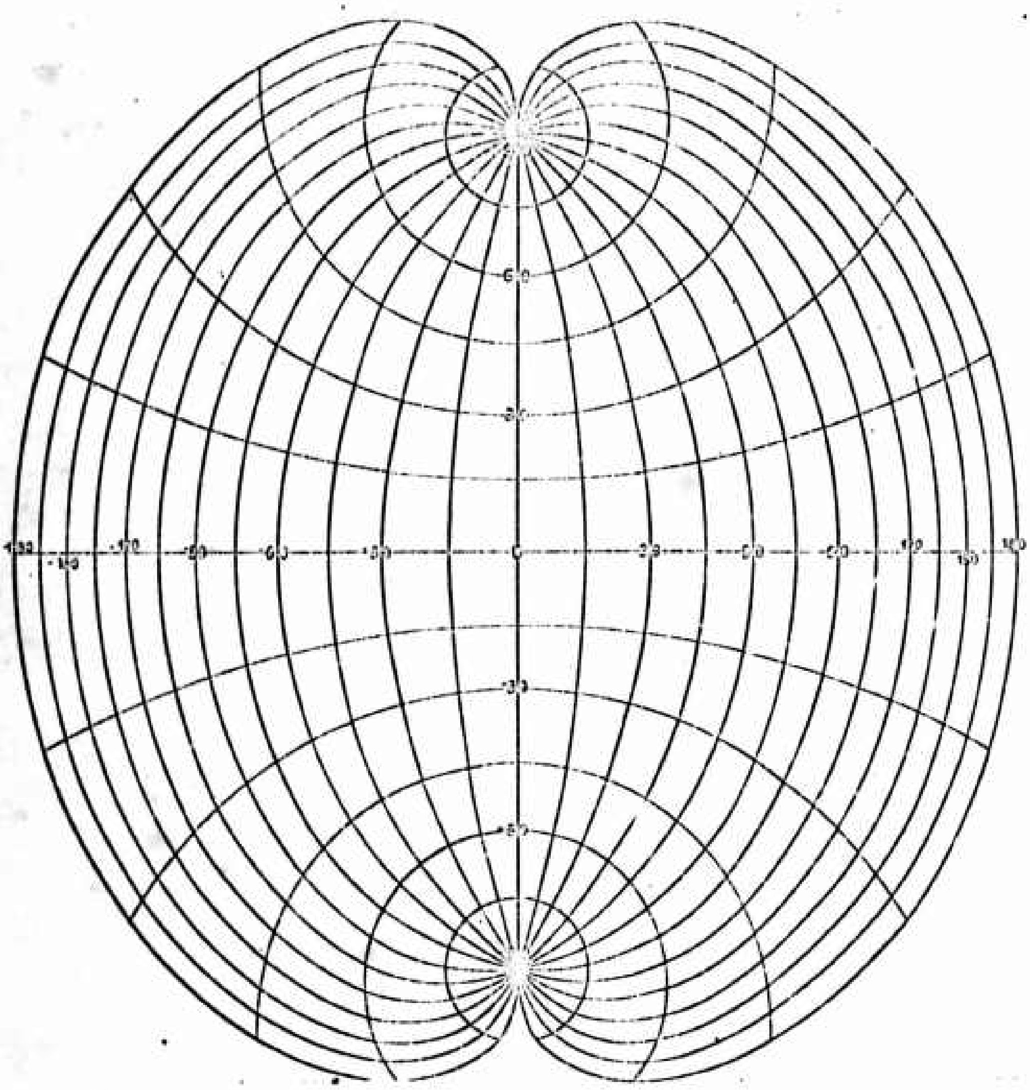


Ill. 9. Shape-true circle-grid (No. 171)

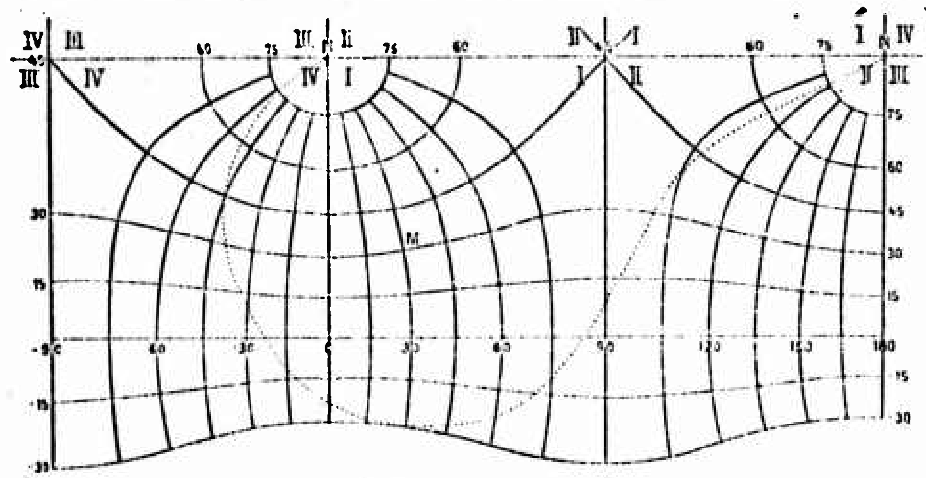


Ill. 12. Area-true projection No. 184 (expanded from circle-grid No. 179)

Plate V



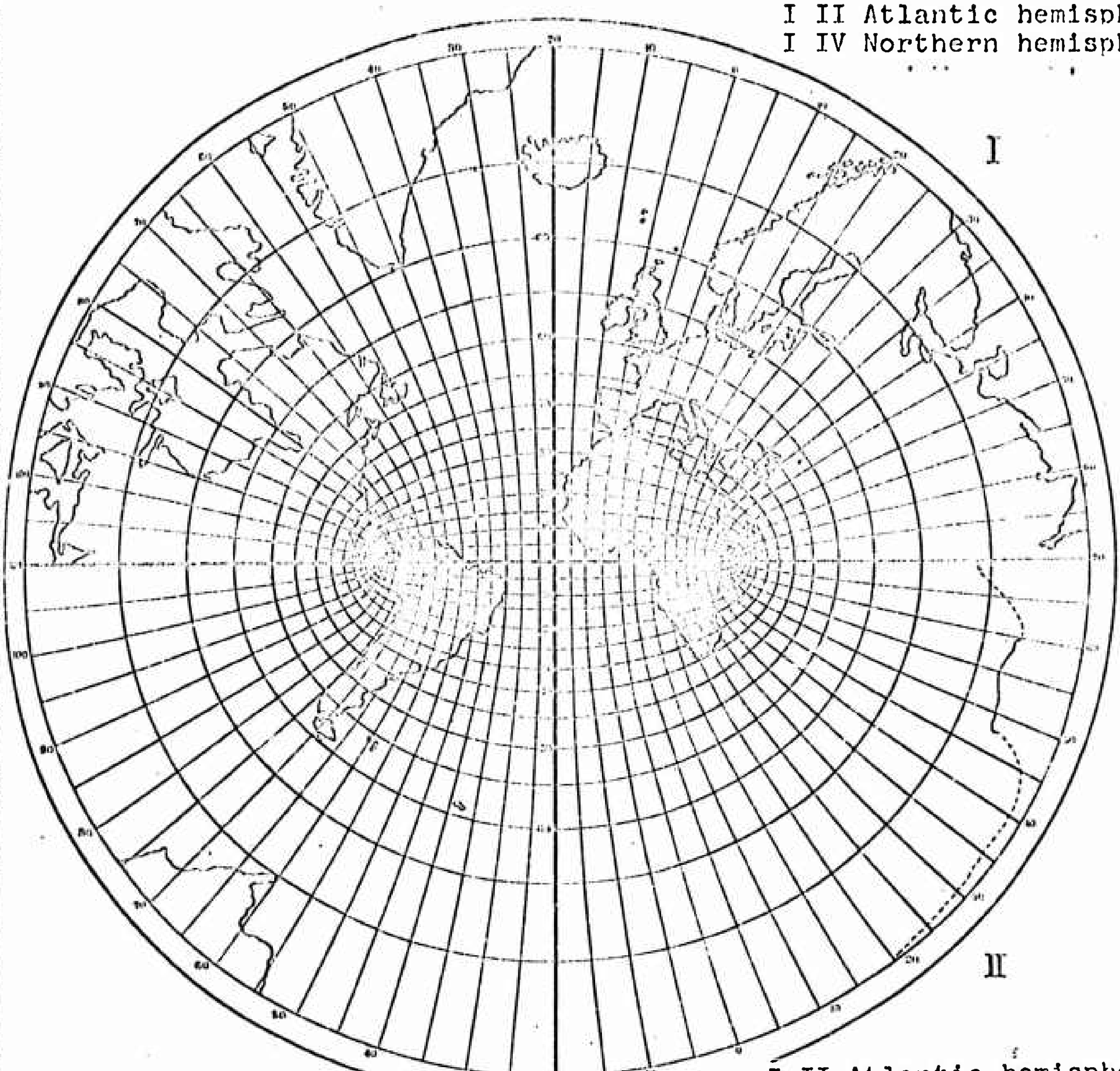
Ill. 10. All-conical area-true circle-grid (No. 179)



Ill. 15. Guyou's shape-true quinconx grid

Plate VIa

I II Atlantic hemisphere
I IV Northern hemisphere



I II Atlantic hemisphere
III II Southern hemisphere

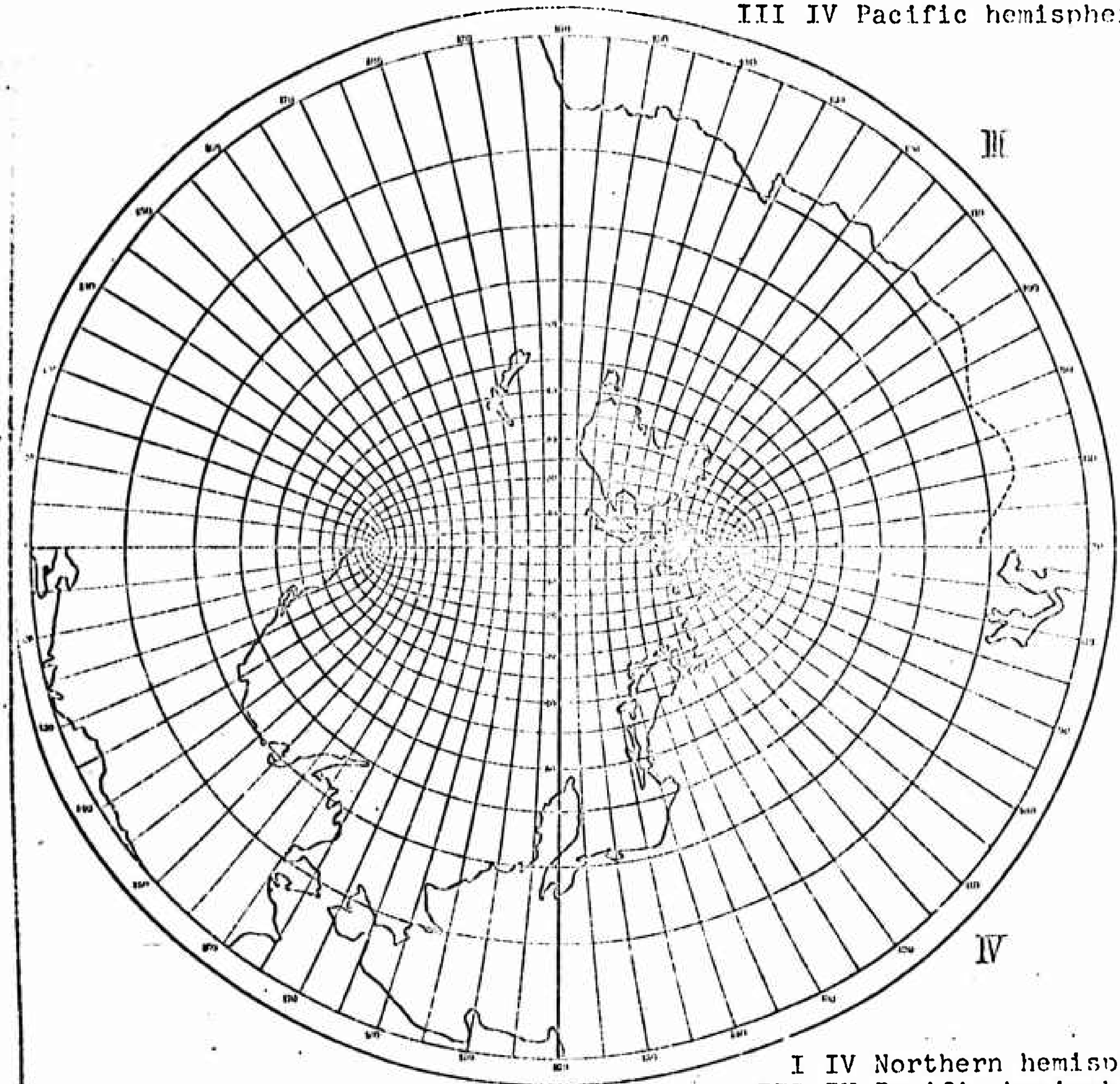
Ill. 13. Shape-true azimuth-equal map (No. 189)

Leitung: Prof. Paul Langhans

GOTHA: JUSTUS PERTHES

Plate VIb

III II Southern hemisphere
III IV Pacific hemisphere



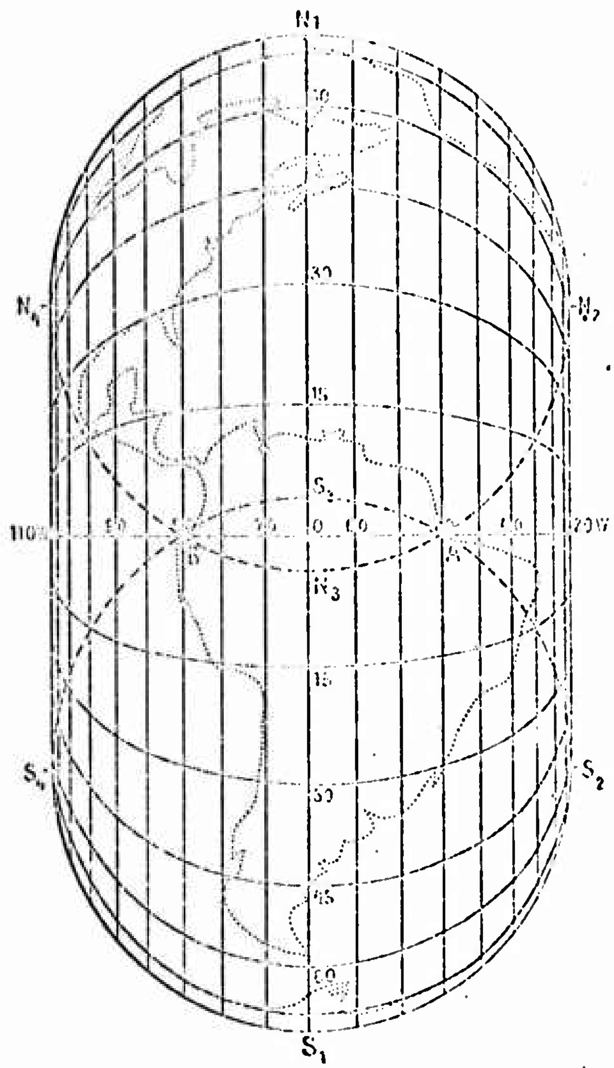
I IV Northern hemisphere
III IV Pacific hemisphere

Ill. 13. Shape-true azimuth-equal map (No. 189)

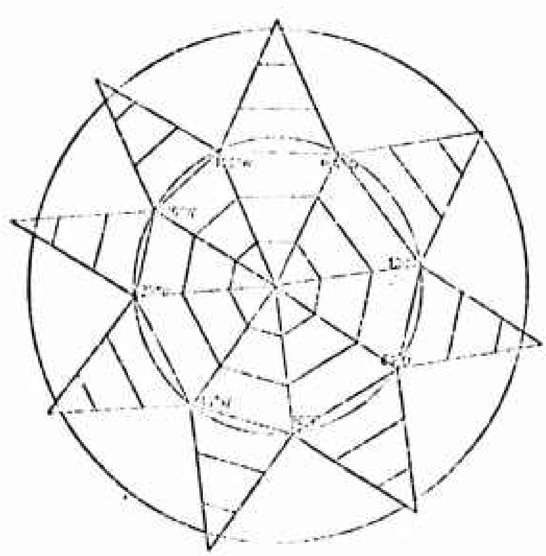
lung: Prof. Paul Langhans

GOTHA: JUSTUS PERTHES

Plate VII

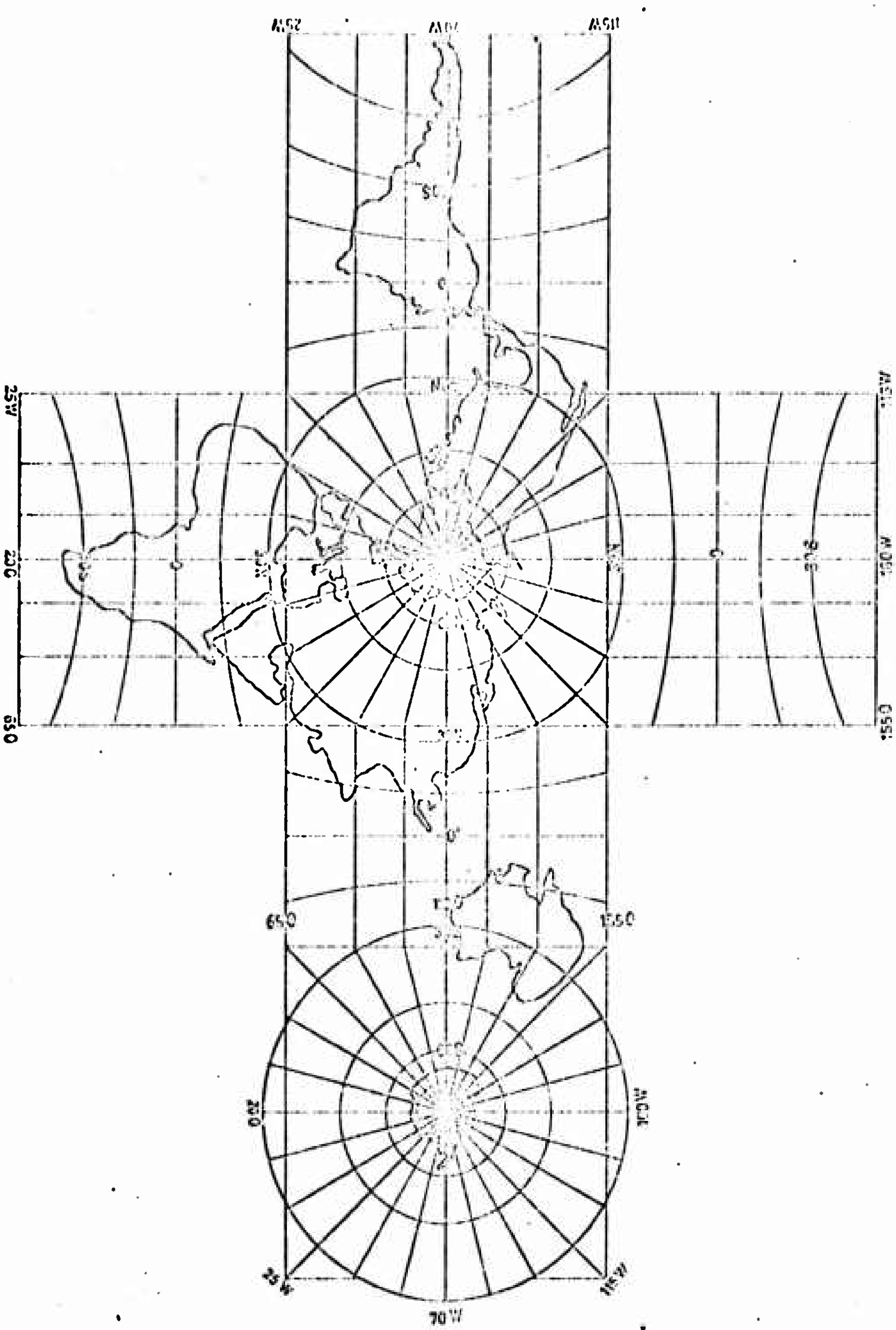


Ill. 16. Maurer's double-counter-azimuthal grid (No. 196)



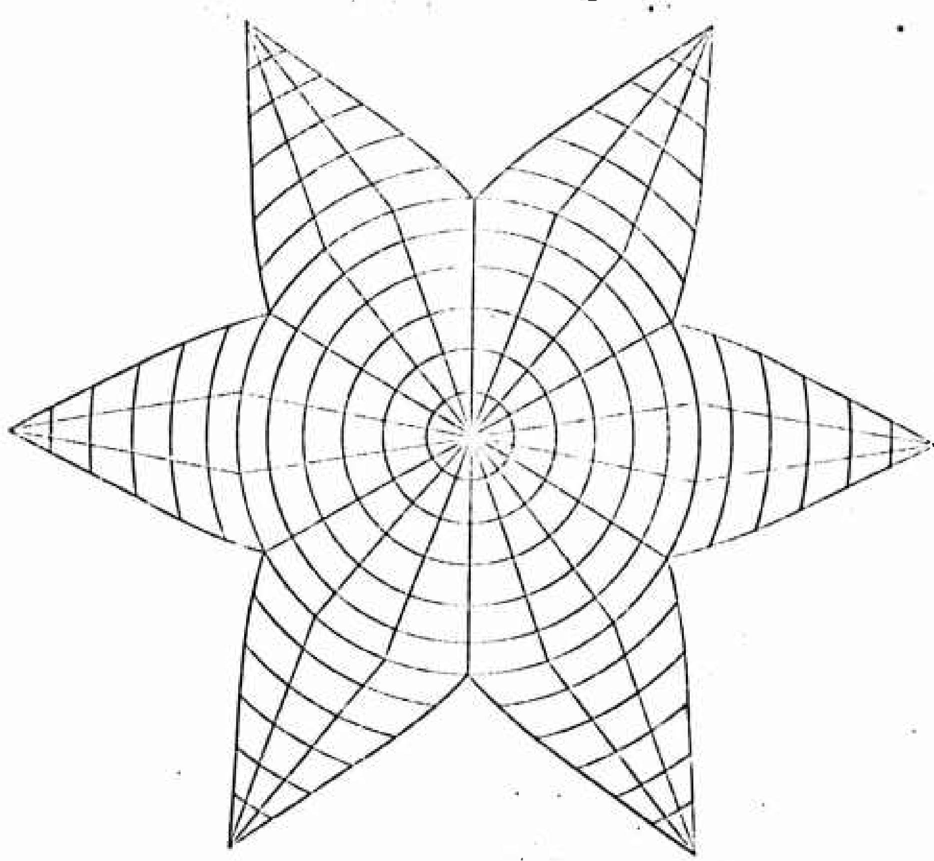
Ill. 20. Jäger's Star-projection (No. 232)

Plate VII (cont.)

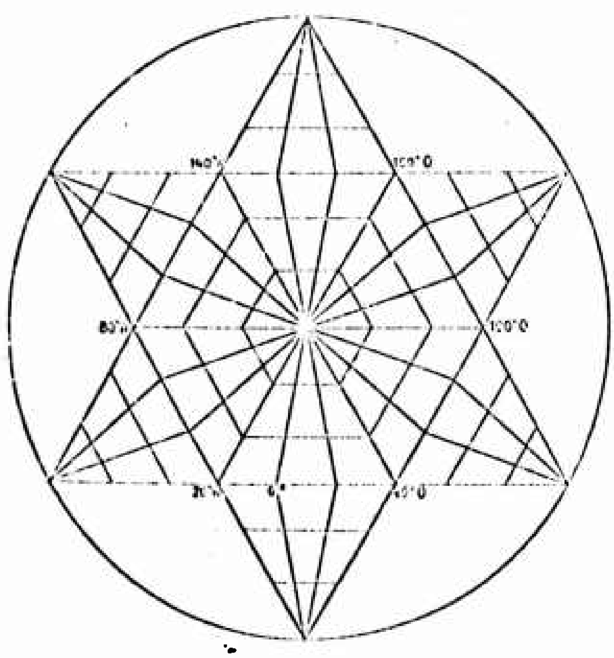


Ill. 24. Center-perspective projection onto eight-cornered box (No. 237)

Plate VIII

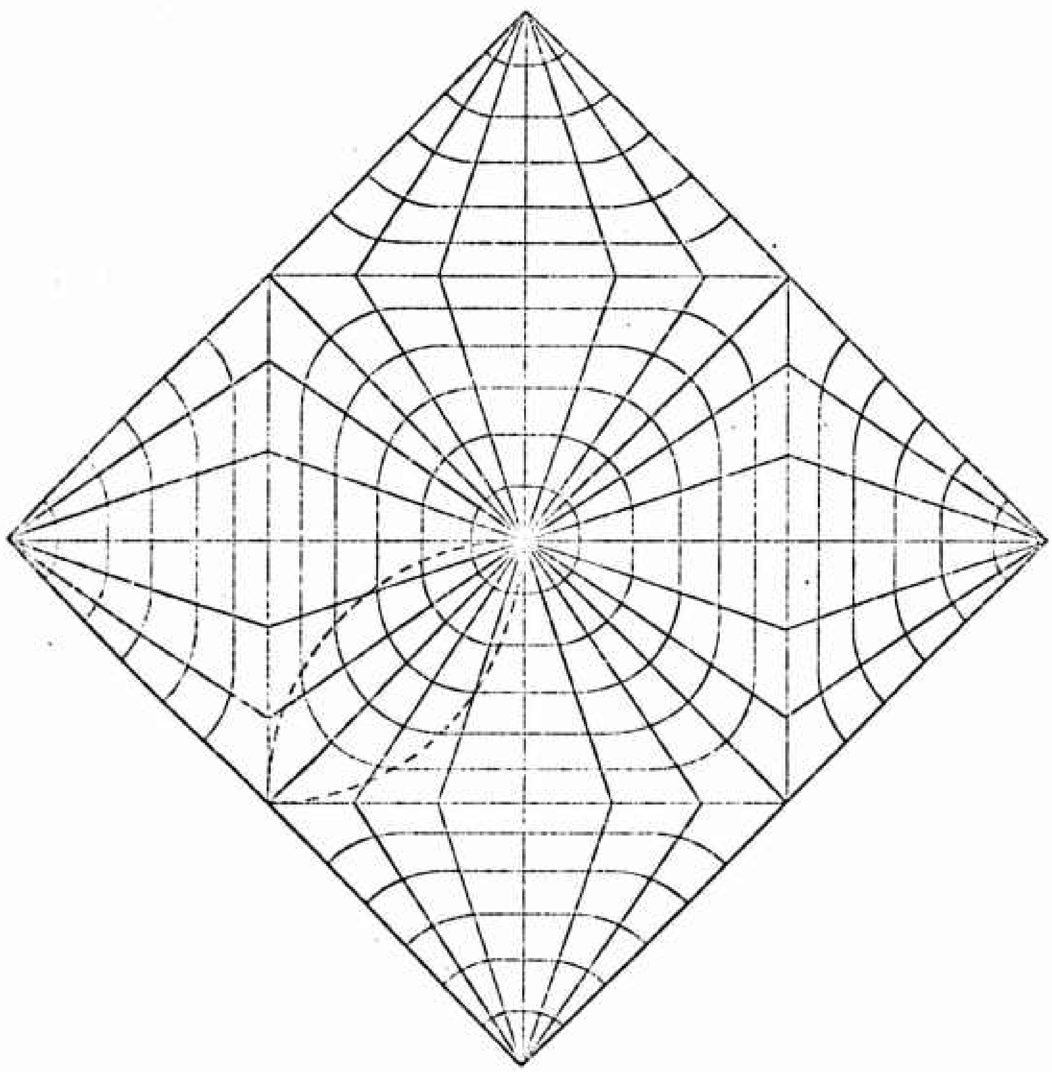


Ill. 19. Maurer's area-true star-projection



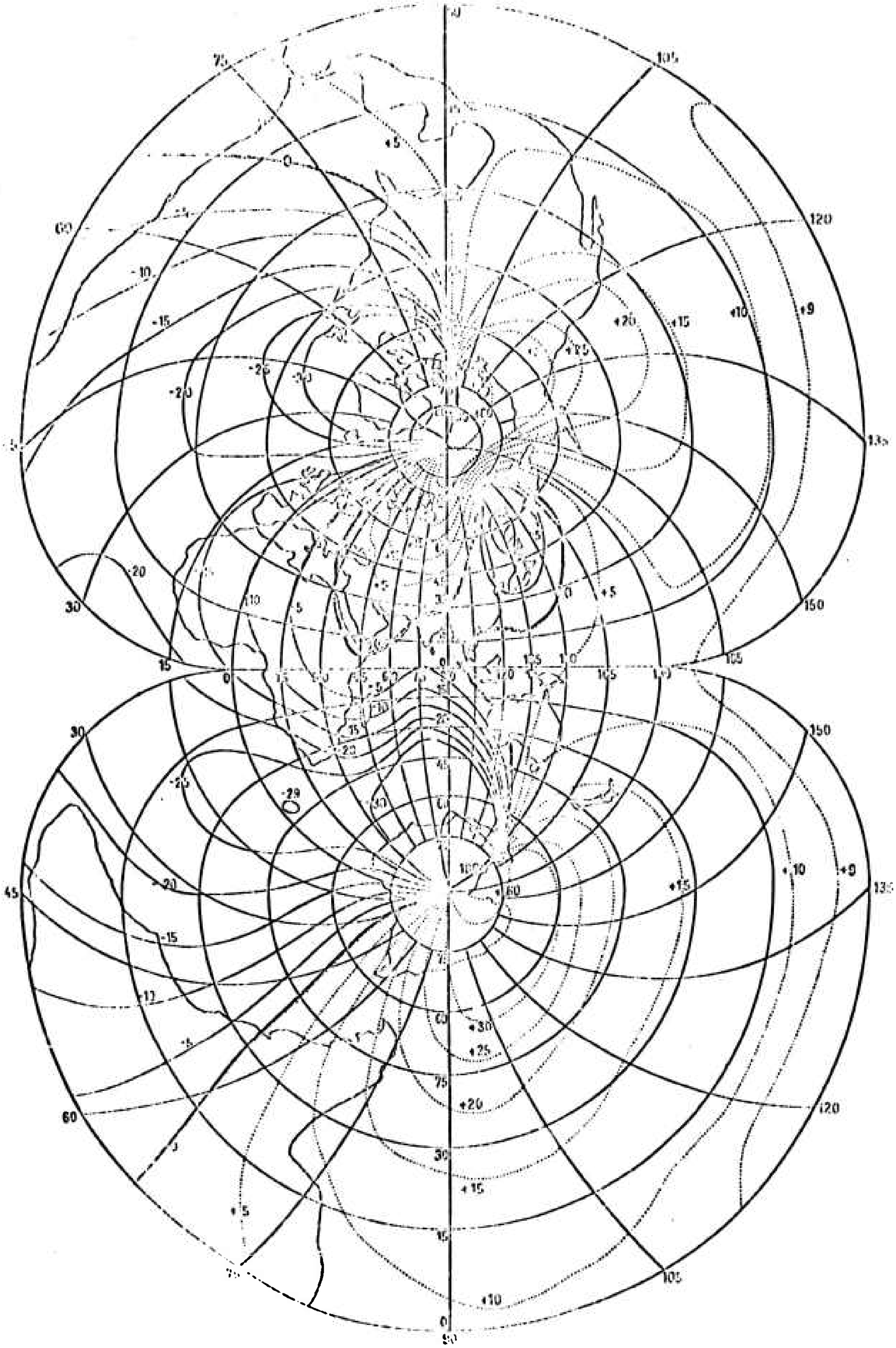
Ill. 21. Simple area-true star projection (No. 233)

Plate VIII (cont.)



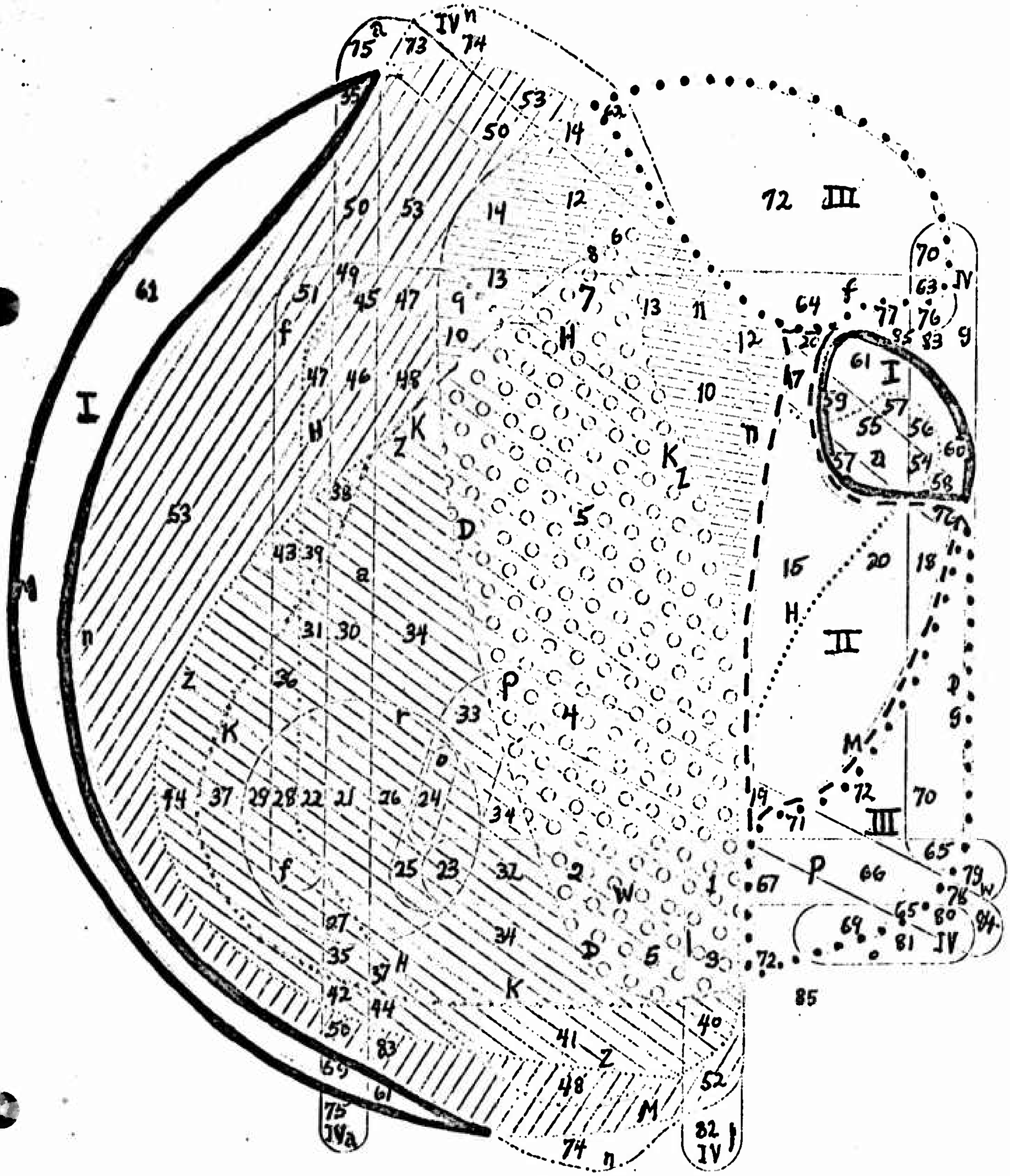
Ill. 23. Zone-true star-projection
No. 235 (quincunx grid)

Plate IX



Ill. 18. Isogone Map (No. 227)

Plate X



Map Symbols

M—— = general, centrally circular
D--- = double symmetrical
n–·– = secondary, circle-spaced
z··· = zenithal (cellular)
K = general conical
r—— = radial
H··· = prime-point map
a—— = true interval

f—— = true area
w—— = true shape
o—— = straight directional
l—— = loxodromic, linear
g—— = counter-azimuthal
p—— = general perspective

Family I, Branch A
" " B
" " $\mathcal{L}$
Family II, Branch A
" " B
" " $\mathcal{L}$
Family III
Family IV

System Table of Map Grids

(H = primary-line, N = secondary-line, E = projection, F = figure, TH = Tissot-Hammer, ZB = Zöppritz-Bludau, H = Herz, G = Gretschel, BF = Bourgeois-Furtwängler)

Family I: True-circular: The N are concentric circles, not straight lines. $r = f(\delta)$

Branch A: Secondary-circle-spaced. All N equal-spaced. $\alpha = g(\delta) \cdot \lambda h(\delta)$

No.	Field No.	Subbranch	Order	Class	Subclass	Group	Sort	Type	Literature
1	29, 65, 80	A. [illegible] All cells permutable	a. [illegible]	I radial azimuthal	a) Prime-point maps	A. Pure-perspective	a. [illegible] perspective -[illegible]	center-perspective (gnomonic) (ca. 600 b.C.) $q = 0$, $c = 1$, $r = tg\,\delta$. Straight-directional for the dromes. [Great-circle-true according to Schols $r = 2\,tg\,\frac{\delta}{2}$ [illegible], $K = 0$] $\Delta' = 90°$	TH 58, ZB 65 (F 41), H 45; G 43, BF 256
2	32, 44, 75		All N straight	$n = 1$	$r(0°) = 0$	$r = (q+c)\sin\delta$	b. [illegible] perspective $q = 1$	All-circular (stereographic) (Hipparch, 160–125 b.C.) $q = c = 1$; $r = 2\,tg\,\frac{\delta}{2}$ shape-true. [Great-circle-true according to Schols $r = 2\,tg\,\frac{\delta}{2}$; [illegible], $K = 1$] $\Delta' = 180°$	TH 59, ZB 67 (F 31); H 47, G 60, BF 257
3		$h(\delta) = n$	$g(\delta) = 0$	$\alpha = \lambda$	shape-true at point $r = \delta = 0$	$q + \cos\delta$	c. outer-perspective	distant perspective (orthographic) (Apolonius, 240 b.C.) $q = \infty$; c arbitrary, $r = \sin\delta$ $\Delta' = 90°$	TH 60, ZB 54, H 46, G 103; BF 256
4			$\alpha = n\lambda$			r maximum	(external) $q > 1$	De la Hire (1704) " " ; $q = (2+\sqrt{2})$ [illegible]; $r(90°) = 2r(45°)$ $\Delta' = 125.9$	65 71; 91
5						for $\delta = \Delta'$ when $\sec\Delta' = -q$, eyepoint below base-circle by q, map-surface above base-circle by c		" " (generalized) c arbitrary; $q = [illegible]$, $r(2\Delta) = 2r(\Delta)$ [illegible]	new
6								Gretschel (?) 1873 $q = 0.5\sqrt{[illegible]} = 1.618$ Length increase the same at $\delta = 0°$ and $= 90°$ $\Delta' = 133°.2$	
									H 75, G 94
7	25							Lowry (1825) $c = 0$, q: mean of the 18 values for which $r(\delta = n\,5°) = n$ [illegible], $n = 1, 2, 3, \ldots 18$ $\Delta' = 126.1$	G 95
8	71							Parent II (1704) $c = 0$, $q = \sqrt{3} = 1.732$ Greatest length increase a minimum $\Delta' = 125°.3$	G 95
9	84							" " $c = 1$, $q = 1$ $\Delta' = 120°$	G 95
10							Stipulations for hemispheres	Parent I (herz) q from $(q+c)(c+1)\frac{dA}{dq} - (1-c)A = 0$, where $A = (q^2-2q^2-2)\cdot 2q^2(q-1)$: entire change of length increase from $\delta = 0°$ to $\delta = 90°$ a minimum. $c = 1$, $q = 1.603$, $\Delta' = 135°.6$	TH 61 H 74; G 94
11								Parent I (herz) as 10, except that $c = 0$; $q = 1.594$ $\Delta' = 118°.8$	H 75; G 94
12								Herz (1885) q from [illegible] where $A = [illegible]$; [illegible] minimum, $c = 1$; $q = 2.133$; $\Delta' = 117°.9$	H 76, G 96
13								Parent III (1704) where ratio of area increase at [illegible] and its highest value K is employed for δ', where $\cos\delta' = \frac{q}{2} - \frac{3}{2q}$; $c = 0$, $q = 2.100$, $\Delta' = 118°.4$	H 77, G 96
14								James (1857) $q = 1.5$; k_1 and k_2 greatest and least ratio of change of length at each point. c is determined that $\sum[(k_1-1)^2 + (k_2-1)^2]$ becomes minimum above the surface of the hemisphere $c = 0.5772$; $\Delta' = 131°.8$	H 209, G 96
15								Clark (1862) - Fiorini (1881) The same minimum stipulation for hemisphere as James No. 14, but q and c are variable $q = 1.470$; $c = 0.548$; $\Delta' = 132°.9$	H 210
16							Stipulations for calottes	Escher (ca. 1850) $r(\Delta) = \Delta_1$; $\sec\Delta' = q$; $0 \sin\Delta_1 - q(\Delta_1 - \sin\Delta_1) = \Delta_1\cos\Delta$, $c = 1$; $q = (\sin\Delta_1 - \Delta_1\cos\Delta) : (-\sin\Delta)$	TH 63
17								" , generalized $r(\Delta_1) = \Delta_1$; $r(\Delta_2) = \Delta_2$; Examples $\Delta_1 = 90°$; $\Delta_2 = 45°$, $q = 1.67$; $c = 0.454$; $\Delta' = 126°.8$ or $\Delta_1 = 135°$; $\Delta_2 = 67°.5$; $q = 1.382$, $c = 0.868$, $\Delta' = 136°.3$	new
18								Hammer (1887) calotte area = picture area. $(q+c)\cos\Delta/_2 = q + \cos\Delta$ $c = 1$; $q = (\cos\Delta/_2 - \cos\Delta) : (1 - \cos\frac{\Delta}{2})$	TH 64
19								" generalized for 2 calottes. $q = 1 + 2\cos\frac{\Delta_1}{2}\cos\frac{\Delta_2}{2}$; $c = 2(\cos\frac{\Delta_1}{2} + \cos\frac{\Delta_2}{2}) - q$. Examples: $\Delta_1 = 90°$, $\Delta_2 = 45°$, $q = 2.307$; $c = 0.455$, $\Delta' = 115°.7$	new
20								Tissot I (1878–80) Length increase at $\delta = 0$ so great as at Δ; c arbitrary; $q = 0.5(1 + \sqrt{5 + 4\cos\Delta})$. Example: $\Delta = 90°$; $q = 1.618$; $\Delta' = 128°.2$	TH 61
21								Tissot II " " Highest value of ratio of area increase K in calotte Δ a minimum. c arbitrary q from $q^3 - q^2 - (2 + 3\cos\Delta)q - \cos^2\Delta - \cos\Delta = 1$. Example: $\Delta = 90°$; $q = 2.148$; $\Delta' = 117°.7$	TH 62

SNo	FNo.	Sub-branch	Order	Class	Subclass	Group	Sort	Type Family I (cont.): true-circular. Branch A - Secondary-circle spaced	Literature
22	26					B Not perspective	a. equally projected	Lidman (1877) First from globe center onto contiguous cone at $\delta=45°$; then parallel to the earth's axis projected onto equator plane. $r=1-\frac{1}{2}tg(45°-\delta)$; $r(90°)=\sqrt{2}$, $r(135°)=\infty$	TM 94
23	22						b. area-true	Lambert (1772) (isospheric, isomeric, named incorrectly after Lorgna); $r=2\sin\frac{\delta}{2}$; $r(180°)=2$	TM 65, ZB61 (F67), N76, G241; BF262
24	21						c. intervening	Mercator (1569) (central-interval-true, named incorrectly after Postel (1581)); $r=\delta$; $r(180°)=\pi$	TM 72, ZB56 (F64), N96; G241; BF264
25 26	26							Breusing (1892) $r=2\sqrt{tg\frac{\delta}{2}\sin\frac{\delta}{2}}$ = geometric mean between shape-true F No.2 and area-true F No.23. $r(180°)=\infty$	TM 93, ZB 59, N [illegible] BF265
27								Airy (1861) Balance of errors. Integral of $[(\frac{h_1}{h_2}-1)^2+(h_1h_2-1)^2]$ a minimum over hemisphere surface. $r=2\cot\frac{\delta}{2}$ lognat $\sec\frac{\delta}{2}+tg\frac{\delta}{2}$; $r(90°)=1.693$ James and Clarke (1862) Integral of $[(h_1-1)^2+(h_2-1)^2]$ a minimum over calotte Δ; $r=2\cot\frac{\delta}{2}$ lognat $\sec\frac{\delta}{2}+2tg\frac{\delta}{2}\cot^2\frac{\Delta}{2}$ lognat $\sec\frac{\Delta}{2}$, $\Delta=\delta=90°$ yields $r(90°)=1.336$	TM 71, N321, G247, BF265
28 29	28				B ring maps $r(0°)\gtrless 0$		d. area-true "	Hammer (1887) $r(0°)>0$; $r=\sqrt{r_0^2+4\sin^2\frac{\delta}{2}}$. All N too great, no N scale-true. $r(90°)>\sqrt{2}$, e.g., $r(0°)=0.5$; $r(90°)=1.5$ Mauriz (1914) $r(0°)<0$, $r=\sqrt{\sin^2 d_m+2\cos d_m-2\cos\delta}$. d_m scale-true. Map applies only for $\delta>\Delta$, where $2\cos\Delta=\sin^2 d_m+2\cos d_m$, e.g. $d_m=90°$, $\Delta=60°$, $r(90°)=1$	N220; G248 TM 69
30	27						b. circle-interval-true	" $r(0°)<0$, $r=\sin d_m-d_m+\delta$. d_m scale-true. Map applies only for $\delta>\Delta$, where $\Delta=d_m-\sin d_m$; e.g. $d_m=90°$, $\Delta=32°.7$, $r(90°)=1$	Act. Math. 1914, II, p.64
31	29						c. neither area nor interval-true	$r=a+\sin\frac{\delta}{2}$	new
32 33 34	33			II Conical oval (...) d_m scale-true γ-half cone angle	A) point maps $r_0=f(0)=0$	A. cone-perspective $r=f(c,q)\sin\delta$ / $f(q+\gamma\cos\delta)$ [illegible]	a. intersecting conical $d_m=2(90°-\gamma)$	General case. No special stipulations concerning q, c. Center-perspective. $q=0$, $r=c\sin\delta:\sin(\gamma+\delta)$; d_m from $\sin(\gamma+d_m)=c\sin\gamma$ Center-perspective and zone-equal. Murdoch II (1759). Area-equal zone between δ_1 and δ_2. $c=\sec\frac{\delta_2-\delta_1}{2}\sqrt{\cos\frac{\delta_2-\delta_1}{2}}$; $r=\sqrt{\cos\frac{\delta_2-\delta_1}{2}}[tg\frac{\delta_1+\delta_2}{2}+tg(\delta-\frac{\delta_1+\delta_2}{2})]$; $q=0$; $\gamma=90°-\frac{\delta_1+\delta_2}{2}$	TM 99 TM 101 TM 102, N87; G244
35 36 37							b. tangential-cylindrical $d_m=90°-\gamma$	General case $c=\mathrm{cosec}\,\gamma$, q and γ arbitrary. Special case of 32 Center-perspective $q=0$, $r=\sin\delta\,\mathrm{cosec}\,\gamma\,\mathrm{cosec}(\gamma-\delta)$ Surface-perspective. Braun's stereographic cone projection (1868) $q=1$, $c=2$; $n=0.5$; $\gamma=30°$; $r=3\sin\delta:[\sin 30°+\sin(30°+\delta)]$	TM 99 TM 103 TM 100; N90
38	34					B zone-perspective	intersecting conical	$r=(c-\cos\delta)\sec\gamma$, $d_m=180°-2\gamma$ scale-true. "Rolled off cone" after Tissot-Hammer. Special case ($\delta_2=0$ and $d_m=\delta_1$) of 45	TM 98
39 40	32					C shape-true	a. tangential-cylindrical b. intersecting conical	Lambert-Gauss (1772 and 1822) $r=tg\,d_m\left[\frac{tg\frac{\delta}{2}}{tg\frac{d_m}{2}}\right]^n$; $n=\cos d_m$; $\gamma=90°-d_m$. Secondary-circle d_m scale-true " " $r=\frac{\sin\delta_1}{n}\left[\frac{tg\frac{\delta}{2}}{tg\frac{\delta_1}{2}}\right]^n$; $n=\frac{lg\sin\delta_1-lg\sin\delta_2}{lg\,tg\frac{\delta_1}{2}-lg\,tg\frac{\delta_2}{2}}$; secondary circles δ_1 and δ_2 scale-true. (Harding's sky map)	TM 77; ZB 31, N 109, G153, BF278 TM 81; ZB 40 (F85), N112; G153
41 42 43	30					D center-interval-true $r=\delta$	intersecting conical	n arbitrary. Calotte Δ area-equal for $\Delta\sqrt{n}=2\sin\frac{\Delta}{2}$; d_m scale-true for $n=\sin d_m:d_m$. Example: $\Delta=90°$; $n=0.8125$, $\gamma=54°.3$; $n\cdot 360°=292°.5$ $n=0.739$, $\gamma=47°35'$, special case of 41, in which $\Delta=d_m$; $n\cdot 360°=266°$, $\Delta=d_m=74°52'$; $c=1.142$ $n=0.5$ Schjerning I (1882) $d_m=108°36'$, $\gamma=30°$, $\Delta=158°27'$	ZB 137 (F119)
44						F area-true	intersecting conical	Lambert (1772) $r=2\sin\frac{\delta}{2}:\sqrt{n}$; scale-true and shape-true d_m, where $\cos^2\frac{d_m}{2}=n$. Equal shape-distortion at δ_1 and δ_2, where $n=\cos\frac{\delta_1}{2}\cos\frac{\delta_2}{2}$. Examples: $\delta_2=0$; $\delta_1=90°$; $d_m=65°32'$; $n\cdot 360°=254°.45$; $n=1:\sqrt{2}$	TM 84-87; ZB 128 (F99) G151; BF271
45 46	37				B ring maps $r(0°)>0$	A zone-perspective "	a. intersecting cone b. tangential-cylindrical	δ_1 and δ_2 scale-true, $n=\cos\frac{\delta_1+\delta_2}{2}$; $c=\cos\frac{\delta_1-\delta_2}{2}:\cos\frac{\delta_1+\delta_2}{2}$; $r=(c-\cos\delta):\sin\frac{\delta_1+\delta_2}{2}$; $\gamma=90°-\frac{\delta_1+\delta_2}{2}$ δ_1 scale-true, $n=\cos\delta_1$; $c=\sec\delta_1$; $r=(\sec\delta_1-\cos\delta):\sin\delta_1$; $\gamma=90°-\delta_1$. Special case $\delta_2=\delta_1$ of 45	TM 98

SNo	File	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
47 48 49	35					B. circle interval-true $\rho = r(d_m) + d - d_m$	a. tangential-conical b. intersecting cone	**Ptolemaeus I** (130 A.D.) simple cone. E. $r = tg\, d_m + \delta$; $n = \cos d_m = \sin \gamma$; d_m scale-true **De l'Isle** (1745) d_1 and d_2 scale-true. $n = \cos\frac{d_1+d_2}{2} \sin\frac{d_1-d_2}{2} : \frac{d_1-d_2}{2}$; $r = \frac{1}{n} \sin\frac{d_1+d_2}{2} \cos\frac{d_1-d_2}{2} - \frac{d_1+d_2}{2} + d$. Greatest shape-distortion at d_3, where $\cos d_3 = n = \sin \gamma$ Example: $d_1 = 70°$, $d_2 = 20°$, $\gamma = 43°14'$, $n \cdot 360° = 246°.6$; $d_3 = 46°46'$ **Murdoch I** (1754) Ring-area between d_1 and d_2 area-equal. $n = \cos\frac{d_1+d_2}{2}$; $r = tg\frac{d_1+d_2}{2} \sin\frac{d_1-d_2}{2} : \frac{d_1-d_2}{2} - \frac{d_1+d_2}{2} + d$; $c = \sec\frac{d_1+d_2}{2} \sin\frac{d_1-d_2}{2} : \frac{d_1-d_2}{2}$; $n \cdot r(d_m) = \sin d_m$	TN 94, ZB 112, N 42, G [illegible], BF 274 TN 95, ZB 115 (F 79), N 101, G 155, BF 275
50	37					C. circle-interval-similar $r = r_0 + md$	intersecting cone	**Murdoch II** Ring area between d_1 and d_2 area-equal. $n = \cos\frac{d_1+d_2}{2}$, $r = (tg\frac{d_1+d_2}{2} \cot\frac{d_1-d_2}{2})\frac{d_1-d_2}{2} + m(d - \frac{d_1+d_2}{2})$, $m = \sin\frac{d_1+d_2}{2} tg\frac{d_1-d_2}{2} (\frac{d_1-d_2}{2})^2$, $c = \sec\frac{d_1+d_2}{2} \cot\frac{d_1-d_2}{2} \cdot \frac{d_1-d_2}{2}$	TN 96; N 87; G 102
51 52	36					D. area-true $r = \sqrt{r_0^2 + \frac{4}{n} \sin^2\frac{d}{2}}$	a. tangential cone b. intersecting cone	With scale-true N $n = \cos d_m$; $r_m = tg\, d_m$; $r_0 = \sec d_m - 1$; $r = \sqrt{r_0^2 + \frac{4}{n} \sin^2\frac{d}{2}}$. Special case $d_2 = d_1$ of 52 Equal length- and shape-distortion at d' and d'', where $\cos d' + \cos d'' = 2 \cos d_m$ **Albers** 1805 with 2 scale-true N d_1 and d_2. $n = \cos\frac{d_1-d_2}{2} \cdot \cos\frac{d_1+d_2}{2}$; $r_0 = \frac{2}{n} \sin\frac{d_1}{2} \sin\frac{d_2}{2}$; $r = \sqrt{r_0^2 + \frac{4}{n} \sin^2\frac{d}{2}}$. Greatest length shortening at d_3, where $\cos d_1 + \cos d_2 = 2 \cos d_3 = 2n$	TN 97; G 147
53 54 55	39 38 41		A. corrector-circular $g(d)$ not constant	radial-circular $\alpha = g(d) \cdot \lambda$	All N are arcs of radius R	A. Prime-point maps $r(0) = 0$	a. All N scale-true b. $R > \frac{\pi}{2}$	**Wiechel** (1879) area-true $\alpha = \lambda + \frac{d}{2}$; $r = 2 \sin\frac{d}{2}$ interval-true $\alpha = \lambda + \beta$, where $\beta = d : 2R$; $r = d$ not interval-true, e.g. $\alpha = \lambda + \beta$, where $\sin\beta = \sin\frac{d}{2} : R$, $r = 2 \sin\frac{d}{2}$	TN 33, ZB 130, N 191, G 145, BF 271 TN 70, N 166 new
56 57 58	42 43 44					B. Ring-maps $f(0) = r_0 > 0$	a. interval-true b. area-true c. neither interval nor area-true	$\alpha = \lambda + \beta$, where $\sin\beta = \frac{r_0 + d}{2R}$; $2R > r_0 + \pi$; $r = r_0 + d$ $\alpha = \lambda + \beta$, where $\sin\beta = r : 2R$; $2R > \sqrt{r_0^2 + 4}$; $r = \sqrt{r_0^2 + 4 \sin^2\frac{d}{2}}$ $\alpha = \lambda + \beta$, where $\sin\beta = r : 2R$, $2R > r_0 + 2$; $r = r_0 + 2 \sin\frac{d}{2}$	" " "
59 60	49 50	Bent cellular $n(d)$ not constant	a. Conical-circular (curve-conical)	I. equal-spaced only tg and d	a) all N sub-similar	A. point-ring-maps $f(0) > 0$	a. circle-interval-true $m = 1$	**Mercator-Bonne** (1584, 1752), $n = 1$. All N scale-true. Area-true. Shape-true on central-N and on N d_m; $n \gtrless 1$. All N and all areas changed in ratio n. Nowhere shape-true. At $n > 1$ the map overlaps the tangential-cone of d_m in zone, where $n \sin d - d > tg\, d_m - d_m$	TN 106, ZB 142 (F 89) N 185, G 159; BF 275
61 62	51 53		Central-N changes and $\alpha = b(d) \cdot \lambda = 0$ $g(d) = 0$	interval-similar	$r(d) = n \lambda \dots$ n = constant		b. circle-interval-similar $m \gtrless 1$	$m n = 1$. Area-true. No N scale-true. Nowhere shape-true. $n = 1$. All N scale-true. Arc-similar at ratio m.	
63 64	45 46					B. prime point-maps $f(0) = 0$	a. center-interval-true $m = 1$, $r = d$	$n = 1$. **Stab** (1514), also named after Werner. Special case $d_m = 0$ of 59. All N scale-true. Area-true Shape-true on central-N. Also Schjerning's map no. IV	TN 107; ZB 144, 148 (F 134), N 174, G 162; BF 279
65	47					central-interval-similar $r = m d$		$n < 1$. Example: $n = 0.5$. Schjerning V; $\alpha = \frac{\lambda}{d} \sin d$, area- and distance-similar, nowhere shape-true. Special case $d_m = 0$ of 60	ZB 200 (F 127)
66 67	48 46						b. central-interval-similar $m \gtrless 1$	$mn = 1$. Area-true, Special Case of 61. Example: Schjerning V with another uniform globe $m = \sqrt{2}$, $n = 1 : \sqrt{2}$, $n = 1 : \sqrt{2}$, $r = d\sqrt{2}$, $\alpha = \lambda \sin d : 2d$ $n = 1$. Distance-true " " " 62, " " " " " " $m = 2$ $r = 2d$; $\alpha = \lambda \sin d : 2d$	" " " - - -
68 69		[The combination of 68 and 69 as one map belongs in Family IV.]			Where length of N is due to world map border $\mu = f(d)$	Prime-point maps $r_0 = 0$	central-interval-true $m = 1$, $d_m = 0$ $r = d$	**Schjerning III** (1904) World-map border 2 full circles which are contiguous at $d = 0$. $r = d$; $\alpha = \frac{2}{\pi} \arccos\frac{d}{\pi}$ for $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$. Scale-true N $d_m = 79°27'$. True-shape at points $\lambda = 0$ or $180°$, $d = 0°$, $180°$ or $79°27'$ **Schjerning II** (1904) World-map border ellipses with equator and central-N as axes. Left half of map. $r = d$, $\alpha = \lambda$ for $d < \frac{\pi}{2}$; $\alpha = 2\beta\lambda : \pi$ for $d > \frac{\pi}{2}$, where $tg^2\beta = (5\pi^2 - 4d^2) : (4d^2 - \pi^2)$ **Schjerning II** (1904) Right half. World-border of graphically determined arcs from the poles to equator-point $r = \pi$; $\alpha = 0$; $\alpha = \lambda$ for $d < \frac{\pi}{2}$; $\alpha = 2\beta\lambda : \pi$ for $d > \frac{\pi}{2}$, where β is graphically determined function of d	ZB 195 (F 123) ZB 193 (F 121) - - - -

TEXT NOT REPRODUCIBLE

Sht.	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
70	51			II. not equal-spaced	a) Length or of N axis to world-map border $\mu r = f(\delta)$	A. Area-true $\int_{\delta_0}^{\delta}$ (pedo-$4\pi\sin^2\frac{\delta}{2}$)	a. Line-ring maps $r_0 > 0$	Nell's E. μ = center from 51 and 59. $r\mu = \pi[r\cos\delta_m - \sin(r - tg\,\delta_m + \delta_m)]$. $N\delta_m$ scale-true; r_0 from $r_0^2\cos^2\delta_m - \cos(r_0 - tg\,\delta_m + \delta_m) = 1 - 4\cos\delta_m + \cos^2\delta_m$	
71								" " Example of 70: $\delta_m = 45°$; $\cos\delta = 0.5303 - r^2 : \sqrt{32} + 0.5\cos(r° - 12°17'.75)$; $\mu° = 150° : \sqrt{2} + \frac{180°}{r°}\sin(r° - 12°17'.75)$; $\alpha = \mu\lambda : (2\pi)$; $r_0 = 0.3709$; $r_{180} = 2.5775$. Scale-true and shape-true at δ_m.	Pet. Mitt. 1890, p. 99
72	47						b. prime-point-maps $r_0 = 0$	" " Examples of 90. $\delta_m = 0$; $\cos\delta = \cos^2\frac{\mu}{2} - \frac{r^2}{?}$; $\mu = 150°[1 - \frac{\sin r}{r}]$: center from 23 and 63. $\alpha = \mu\lambda : (2\pi)$; $r_{180} = 2.1971$	
73								Maurer. World map border of 3 semi-circles. $\cos\delta = \frac{2}{3\pi}(\mu\cos\mu - \sin\mu) - \frac{1}{3}$; $r = 4\sin\frac{\mu}{2} : \sqrt{3}$; $x = \mu\lambda : (2\pi)$; $r_{180} = 2.3094$. Scale-true $\delta_m = 45°46'$. Shape-true at $\lambda = 0°$ or $180°$, $\delta = 0°$ or $45°46'$	new

FAMILY I: TRUE-CIRCULAR

BRANCH B: not secondary-circle-spaced

Sht.	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
74	55	A. Radial-circular $\alpha : \pi$ to $\lambda : \pi$		I. All N straight $\alpha = g(\lambda)$		A. prime-point maps $r_0 = 0$	a. interval-true	$r = \delta$; $\alpha = \pi\sin\frac{\lambda}{2}$. True-shape on curves $2\sin r = (\pi^2 - \alpha^2)$	
75	57						b. not interval-true	$r = \sin\frac{\delta}{2}$; $\alpha : \pi = \sin\frac{\lambda}{2}$. True-shape on curves $r = \sqrt{1 - \frac{\pi}{2}\cos\frac{\lambda}{2}}$ or $\pi\cos\frac{\lambda}{2} = 2\cos^2\frac{\delta}{2}$	Pet. Mitt. 1914, II, p. 64, No. II
76	61	is not $\alpha = \lambda$				B. Line-ring maps	" "	$r = a + \sin\frac{\delta}{2}$; $\alpha = \pi\sin.\frac{\delta}{2}$ $\quad \pi\cos\frac{\lambda}{2} = (\sin\delta\cos\frac{\delta}{2}) : (a + \sin\frac{\delta}{2})$	
77	55			II. The N are curves		prime-point-maps	a. interval-true	Maurer, 1919. Double-interval-true from points $\varphi = 0$; $\lambda = \pm L$. $y = (\delta_1^2 - \delta_2^2) : 4L$; $x = \sqrt{\frac{\delta_1^2 + \delta_2^2}{2} - L^2 - y^2}$, where $\cos\delta_1 = \cos\varphi\cos(L + \lambda)$; $\cos\delta_2 = \cos\varphi\cos(L - \lambda)$ Field No. applies for grid of points $\varphi = 0$; $\lambda = \pm L$	Ann. d. Hyd. 1919, p. 77;
78	54						b. centro-azimuthal direction-point	Hammer. Interval-true $r = \delta$; $y = -\delta\sin A$; $x = -\delta\cos A$, where $\cos\delta = \sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos\lambda$ and $\cot A =$ $tg\,\varphi\cos\varphi_0\,\mathrm{cosec}\,\lambda - \sin\varphi_0\cot\lambda$	Deetz and Adams, p. 164 Pet. Mitt. 1910, p. 153; ZB 220
79	56						$(\lambda = 0; \varphi = \varphi_0)$	True-shape and field no. 5? apply for direction-point $\lambda = 0$, $\varphi = \varphi_0$. Field no. 82 has the polar grid, field no. 69 for $\varphi_0 = 0$. r, as in all-circular E no. 2, not true-shape. $r = \frac{1}{2}tg\frac{\delta}{2}$; $y = x\,tg\,A$; $x^2 = r^2 - y^2$. δ and A as no. 78. Field no. 56 applies for grid of the direction-point. Polar grid as no. 78.	
80								r as in area-true E no. 23, zone-true, not area-true. $r = 2\sin\frac{\delta}{2}$, $y = x\,tg\,A$; $x^2 = r^2 - y^2$. δ and A as no. 78. Field no. 56 applies for grid of the direction-point. Polar grid as no. 78.	ZB 221 (F151)
81	59	B. annual circular		Ring maps		Interval-true	All N are straight lines	$2N$, δ_1 and δ_2, scale-true; $\delta_1 = \delta_m + E$; $\delta_2 = \delta_m - E$. Simplified cone E after De l'Isle (1745). (incorrectly called Mercator's E); $r_1 = f(\delta_1)$; $r_2 = f(\delta_2)$, $\alpha_1 = \lambda\sin\delta_1 : r_1$ at $N\delta_1$	ZB 120, BF 275 footnote
82						$f(\delta) = tg\,\delta_m + \delta - \delta_m$	All N are arcs	Ptolemaeus II (130 A.D.) $\delta_1 = 27°$, $\delta_2 = 66°10'$; $\delta_3 = 106°30'$ $\quad \alpha_2 = \lambda\sin\delta_2 : r_2$ at $N\delta_2$	n 197; 6158

FAMILY II: STRAIGHT-SYMMETRICAL. All N are parallel straight lines. The Grid is symmetrical to the central-H as well as to the base-line. $x = F(\varphi) = -F(\varphi)$; $y(\lambda/\varphi) = -y(-\lambda/\varphi)$

BRANCH A: secondary-circle-spaced : $y = \lambda \cdot h(\varphi)$

SNo	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
83	8	A. straight-cylinder $h(\varphi)=n \cdot R$ (R = radius of the cylinder) All there parallel straight lines	I. cylindrical (not cylindrical maps)	a) line-sim maps $F(\varphi\varphi)$	A circle perspective $y = R\lambda = n\lambda$	a. contiguous-cylindrical $R = n = 1$		$y=1$, $a=2$. Braun's "stereographic-cylinder E" (1867) $N(\varphi=0)$ scale-true and shape-true. $x = tg\frac{\varphi}{2}$; $y=\lambda$. 1 N shape-true	TH 31 H 91
84								$y=0.4$; $a=1.4$ " "modified Mercator E" (1567). $N(\varphi=0)$ scale- and shape-true; $\varphi_m = 65°41'.6$ shape-true. $x = 1.4 \sin\varphi : (0.4 + \cos\varphi)$; $y = \lambda$; 2 N shape-true.	TH 31
85				cylinder-circle $x = F(\varphi)$ and cylinder-radial $y = n\lambda$	$R = \varphi : (a-1)$	b. intersecting-cylindrical $R = n = \cos\varphi_m$		$y=1$, $a = (1+\cos\varphi_m) : \cos\varphi_m$ at stereographic intersecting cylinder E. $N(\pm\varphi_m)$ scale- and shape-true. $x = (1+\cos\varphi_m) tg\frac{\varphi}{2}$; $y = \lambda\cos\varphi_m$. 3 N shape-true.	NEW
86						$n<1$ c. free-cylindrical $n>1$		$y=0.4$; $R=n=1.04$, $a=1.3346$. Maurer's circle-perspective E. 4 N shape-true $\varphi_m = \pm 22°57'.5$ and $\varphi_m = \pm 67°4'.1$. $x = 1.44 \sin\varphi : (0.4+\cos\varphi)$; $y = 1.04\lambda$. 4 N shape-true	"
87	7				B. area-true $x = \sin\varphi$; $y = n\lambda$	a. contiguous-cylindrical $n=1$		Lambert (1772), zone perspective $x = \sin\varphi = \lambda$. $N(\varphi=0)$ scale- and shape-true. Special case $\varphi_m = 0$ of 55.	TH 27; ZB 159 (F97), H 154, G 113; BF 264
88						b. intersecting-cylindrical		$x = \sin\varphi \sec\varphi_m$; $y = \lambda\cos\varphi_m$. $N(\pm\varphi_m)$ scale- and shape-true. Equal shape-distortion at φ_1 and φ_2, where $\cos\varphi_1 \cos\varphi_2 = \cos^2\varphi_m$	TH 26; ZB 160 BF 267
89						$n = \cos\varphi_m$		Behrmann (1910) $\varphi_m = 30°$; $x = 2\sin\varphi : \sqrt{3}$; $y = \lambda\sqrt{3} : 2$. Least average uppermost shape-distortion on the world map	Act. M.H. 610, II, plate 27
90	6				C. circle-intersection-true	a. contiguous-cylindrical $n=1$		Quadratic flat map, Thoene (1527) $x = \varphi$, $y = \lambda$. $N(\varphi=0)$ shape-true	TH 28, ZB 155 (F95); H 95; G 111, BF 264
91					$x = \varphi$	b. intersecting-cylindrical $n \gtrless 1$		Rectangular " " Marinus of Tyrus (100 A.D.) $x = \varphi$, $y = \lambda\cos\varphi_m$; $N(\pm\varphi_m)$ shape-true. $n = \cos\varphi_m$	TH 25, ZB 57; H 106, G 112; BF 269
92	5	Also No. 92, FNo 5 and family II apply only for $q=0$, family III and FNo. 33 (SNo. 147) apply for $q \gtrless 0$.		b) prime-point maps	A. cylinder perspective	c. intersecting-cylindrical $n = \cos\varphi_m$		q arbitrary. $x = \cos\varphi_m [\sin\varphi + q(1-\cos\varphi)] \sec\varphi$; $y = \lambda\cos\varphi_m$; $N(\pm\varphi_m)$ scale-true, $N(\varphi=0)$ shape-true. FNo 5 and fam. II for $q=0$. Family III, SNo 147, FNo 33 apply for $q \gtrless 0$.	TH 29
93	5			$F(\pm 90°) = \infty$	Sighting point q below base-circle	b. contiguous-cylindrical $n=1$		$q=0$. Wetch centre-perspective $x = tg\varphi$; $y = \lambda$. $N(\varphi=0)$ scale- and shape-true. Special case $(\varphi_m = 0)$ of 93.	TH 30; H 124; G 131
94	1				B. shape-true not perspective	course-linear (loxodromic)		Etzlaub (1511) - Mercator (1569) (Mercator projection or navigation map). Contiguous-cylindrical $x = \log nat\, tg(45° + \frac{\varphi}{2})$; $y = \lambda$. $N(\varphi=0)$ scale-true. Loxodromes: straight lines	TH 21, ZB 163 (F99), H 114; G 119; BF 267
95								" " " " intersecting-cylindrical $x = \cos\varphi \log nat\, tg(45° + \frac{\varphi}{2})$; $y = \lambda\cos\varphi_m$. $N(\pm\varphi_m)$ scale-true. Loxodromes: straight lines	
96	3				C. not shape-true	a. course-linear		Intersecting-cylindrical, Etzlaub Mercator affine transformed $x = m \log nat\, tg(45° + \frac{\varphi}{2})$; $y = n\lambda$, $m \gtrless n$	new
97	5				not perspective	b. not course-linear		" " $x = \sqrt{tg\varphi}$; $y = n\lambda$. $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = n$	new
98	9	B. not cylinder ... cylindrical	a. prime-point maps $F(\pm 90°) = \infty$	I. area-true $y = \lambda\cos\varphi \frac{dx}{d\varphi}$		$x = n\, tg\varphi$; $y = \frac{\lambda}{n}\cos^2\varphi$		Type according to value of n; $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = \sqrt{n}$. Shape-true points $\lambda = 0$, $\varphi = \pm\varphi_m$; H = algebraic curves $y(n^2 + x^2)^{1/2} = n^2\lambda$ with asymptote $y = 0$	New
99	10	H are curves		II. All N scale-true		$x = n\, tg\varphi$; $y = \lambda\cos\varphi$		" " " " " . All N scale-true. Shape-true points $\lambda = 0$, $\varphi = \pm\varphi_m$, where $\cos\varphi_m = \sqrt{n}$. H = algebraic curves $y(n^2 + x^2)^{1/2} = n\lambda$ with asymptote $y = 0$	"
100	11		b. Ring maps $F(\pm 90°)$ finite	I. central of equal-spaced $x = m\varphi$	a) pointing maps $m \gtrless 2$	A. All N are scale-similar $y = n\lambda\cos\varphi$	a. interval-true $m=1$		$n=1$ Mercator's (1606) area-true E (usually named after Sanson (1650) and Flamsted (1701)). True-shape at central N and base-line. Earth-areal. Distance-true	TH 93; ZB 164 (F100); H 159, G 160; BF 276
101	12							$n \gtrless 1$ area- and distance-similar, nowhere shape-true. Least shape-distortion on hemisphere for $2m^2 = \sqrt{4 + \pi^2}$	
102	13			interval-similar	$y = 0$ for $\varphi = \pm 90°$	area-similar	b. interval-similar	$mn = 1$ area-true. No N scale-true, nowhere true-shape	TH 42
103	14					N = sine lines	$m \gtrless 1$	$n = 1$. All N scale-true. Area-similar, nowhere true-shape.	

SNo	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
104	14	See footnote p.		re. No. 104-106		B. not all N scale-similar but on base-line	a. Each N a pair of straight lines each through a pole and dividing point on base-line	General case (m arbitrary). $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = m(\pi-2\varphi_m):\pi$. Shape-true at $\varphi=\lambda=0$. Area-equal globe-biangles, if $m=2:\sqrt{\pi}$	
105	12						$y=m2(\pi-2\varphi):\pi$	Special case $m=1$. Interval-true. Base-line and central-N scale-true. Shape-true on base-line $x=0$.	new
106	14					$(x=0)$ $y=m\lambda$		Special case $m=2:\sqrt{\pi}$. Area-equal globe-biangles between the N. $N(\varphi_m=\pm 12°0)$ Shape-true on base-line $x=0$	new
107	14						b. Each N a semi-ellipse through the 2 poles and dividing point on base-line $y=\frac{m\lambda}{\pi}\sqrt{\pi^2-4\varphi^2}$	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = \frac{m}{\pi}\sqrt{\pi^2-4\varphi_m^2}$. Shape-true on base line $x=0$. Area-equal globe-biangles, if $m=\sqrt{8}:\pi$	new
108	12							Special case $m=1$. Interval-true. Base-line and central-N scale-true. Shape-true on base-line $x=0$. Apianus II (1524), used by Bonne	TH 52; Wagner
109								Special case $m=\sqrt{8}:\pi$. Area-equal globe-biangles between the N; $N(\varphi_m=\pm 33°2)$ scale-true. Shape-true on base line $x=0$	
110					b) line-wing maps	A. combination maps between No. 90 $(x=m\varphi;\ \varphi=m)$ and sub-class	y from No. 101–103 for $n=m$ $y=\frac{m\lambda}{2}(1+\cos\varphi)$	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = m:(2-m)$. Shape-true on base-line. Area-equal globe-biangles, if $m=2:\sqrt{\pi+2}$	
111	12						N sine-lines	Special case $m=1$. Winkel I (1921) Interval-true. Scale-true and shape-true on base-line $x=0$	Geog. Zeitschr. 1921, p. 248
112	14					100 to 109		Special case $m=2:\sqrt{\pi+2}$. Eckert V. Area-equal globe-biangles betw. the N. $N(\varphi_m=\pm 37°9)$ scale-true. Shape-true on base line.	ZB 1:2 (F 142)
113	14	"	"	re. No. 113–115			y, from No. 101–103 $y=m\lambda(\pi-\varphi):\pi$	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = m(\pi-\varphi):\pi$. Shape-true at points $\lambda=0$, $\varphi=0$ and $\varphi=\pm 37°33$. Area-true globe-biangles, if $m=\sqrt{8}:(3\pi)$	
114	12					$y=\frac{1}{2}(m\lambda+y_1)$		Special case $m=1$ Eckert I. Interval-true. Scale-true: central-N and 2 N, $\varphi=0$, $\varphi=\pm 37°33$. Shape-true as No. 46	ZB 203 (F 130)
115	14					Not more than 3 scale-true	N: straight-line-pairs	Special case $m=\sqrt{8}:3\pi$. Area-equal globe-biangles betw. the N. Scale-true $N(\varphi_m=\pm 47°1)$ " " "	
116	14					$N(\pm\varphi_m$ and $\varphi=0)$	y, from No. 107–109 $y=\frac{m\lambda}{2}\left[1+\frac{\sqrt{\pi^2-4\varphi^2}}{\pi}\right]$	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = \frac{m}{2\pi}\sqrt{\pi^2-4\varphi^2}$. Shape-true on base line. Area-equal globe-biangle, if $m=4:\sqrt{\pi}(4+\pi)$	
117	12							Special case. Interval-true. Scale- and shape-true on base-line.	
118	14						N: semi-ellipses	Special case. $m=4:\sqrt{\pi}(4+\pi)$ Eckert III. Area-equal globe-biangles betw. the N. Scale-true $N(\varphi_m=\pm 35°0)$. Shape-true as 116	ZB 206 (F 134)
119	14					B. Combination maps between 41	y, from No. 101–103	General case (m and n arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = \frac{2}{\pi}n\cos\varphi'$. Shape-true at $\lambda=0$, $\varphi=\varphi_m$, where $\cos\varphi_m = (n\cos\varphi'):(2m-n)$. Area-true globe-biangles, if $mn=4(2+\pi\cos\varphi')$	
120	12					$x=m\varphi;\ y=n\lambda\cos\varphi'$ and sub-class 100 to 109	$x=m\varphi,\ y=n\lambda\cos\varphi'$ $y=\frac{n\lambda}{2}(\cos\varphi'+\cos\varphi)$	Special Case $m=n=1$ Winkel (1921) Interval-true. Scale-true $N(\pm\varphi')$. Shape-true at $\lambda=0$; $\varphi=\pm\varphi'$	Geog. Zeitschr. 1921, p. 248
121	14							Special case $m=n=2:\sqrt{2+\pi\cos\varphi'}$ Area-true globe-biangles betw. the N. Scale-true $N(\pm\varphi_m)$, where $\cos\varphi_m=\sqrt{2+\pi\cos\varphi'}-1$; Shape-true at $\lambda=0$, $\varphi_m=\pm\varphi'$	
122	12					$y=\frac{1}{2}(n\lambda\cos\varphi'+y_1)$ At most 2 scale-true N		Special case, No. 120 and 121 combined. $\cos\varphi'=\frac{2}{\pi}$. $\varphi'=50°27'.6$. Interval-true, and area-equal globe-biangles. Scale-true central-N and $N(\varphi_m=\pm 50°27'6)$ Shape-true at $\lambda=0$; $\varphi_m=\pm 50°27'.6$	
123	14							Special case of 119. $n=1$; $m=4:(2+\pi\cos\varphi')$; $\varphi'=\pm 59°43'1$; $m=0.753$; area-equal globe-biangles. Equal shortening of central-N and base-line. Scale-true $\varphi_m=\pm\varphi'$. Shape-true at $\lambda=0$; $\varphi=\pm 65°50'.14$.	Work of K.H. Wagner, p. 29
124	14					C. with 4 scale-true N $(\pm\varphi_1$ and $\pm\varphi_2)$	every N a pair of straight lines through dividing points of scale-true N of a hemisphere	General case $x=m\varphi$; $y=\frac{\lambda}{\varphi_2-\varphi_1}[(\varphi_2\mp\varphi)\cos\varphi_1-(\varphi_1\mp\varphi)\cos\varphi_2]$	
125	12							Special case $m=1$. F by Denis (1482) and Donny (1849) $x=\varphi$; y (as 124), interval-true	TH 45 H 108
126	14							Special case $m=\frac{2}{\pi}(\varphi_2-\varphi_1):[(45°-\varphi_1)\cos\varphi_2-(45°-\varphi_2)\cos\varphi_1]$. Globe-biangles area-equal. Example: $\varphi_1=30°$; $\varphi_2=60°$; $x=1.364\varphi$; $y=\lambda(1.232-0.491\varphi)$	
127	14					D. N-scale acc. to equation $y=n\lambda\cos^c p\varphi$ c = whole no. $0<p<1$	$c=1$ $p=2/3$ $y=n\lambda\cos\frac{2}{3}\varphi$	General case (m arbitrary). Scale-true $N(\pm\varphi_m)$, where $\cos\varphi_m = n\cos\frac{2}{3}\varphi_m$. Shape-true at $\lambda=0$; $\varphi=\pm\varphi_m$	
128	12							Special case $m=1$ K.H. Wagner Interval-true. Wagner's example: $\varphi_m=\pm 40°$; $n=0.587$.	K.H. Wagner, p. 9
129	14							Special case $mn=\frac{2}{3}\operatorname{cosec}\frac{\pi}{3}=0.7695$. Globe-biangles area-equal	

TEXT NOT REPRODUCIBLE

SNo	FNo	Sub-branch	Order	Class	Subclasses	Group	Sect	Type	Literature
130	12	cf. footnote p						Special case of 129. $m=1$, $n=1.7695$. Globe-bangles area-equal and interval-true E. Scale-true central H and $N(\varphi_m = \pm 49°57')$	
131		of No 132		Central H and ... equal-spaced	... $\int y \frac{dx}{d\varphi} d\varphi$; $y = y(\lambda)$	lengths of K acc. to prescribed world map border	F(x) acc to 115 Line-ring maps	F(x) acc. to 112, H: sine-line Eckert VI (1906) $m = 2:\sqrt{\pi+2}$, x from $\sin\varphi = \frac{\pi}{2}(x + m\sin\frac{x}{m})$, $y = \frac{m\lambda}{2}(1+\cos\frac{x}{m})$.	
132								Scale-true on $N(\varphi_m = \pm 49°11')$ Shape-true at $\lambda=0$; $\varphi = \pm 49°11'$. F(x) acc to 115, H: pairs of straight-lines. Eckert II (1906) $m = \sqrt{8}:3\pi$; $x = m\pi - \sqrt{m^2\pi^2 - 2\pi\sin\varphi}$; $y = \lambda(m - \frac{2x}{\pi})$.	ZB 214
133								Scale true $N(\varphi_m = \pm 55°10')$ Shape true at $\lambda=0$; $\varphi = \pm 55°10'$. F(x) acc to 115, H: semi-ellipses Eckert IV (1906) $m = 4:\sqrt{\pi(4+\pi)}$; $x = \frac{m\pi}{2}\sin\theta$, $y = \frac{m\lambda}{2}(1+\cos\theta)$, where $(4+\pi)\sin\varphi = 4\sin\theta + \sin 2\theta$. Scale-true $N(\varphi_m = \pm 40°29'$. Shape-true at $\lambda=0$; $\varphi = \pm 40°29'$	" 204 BF 379; " 209 (F136) BF 280
134		cf. footnote p of No 134				$y\pi = F(x)$; $\pi\sin\varphi = \int_0^x F(x)dx$	D F(x) acc to 124-129 pointing maps	F(x) acc. to 122, H pairs of straight lines Collignon (1865) $m = 2:\sqrt{\pi}$; $x = \sqrt{\pi}(1 - \sqrt{2}\sin(\frac{\pi}{4} - \frac{\varphi}{2}))$; $y = \lambda\frac{2}{\sqrt{\pi}}\sin(\frac{\pi}{4} - \frac{\varphi}{2})$	
135								Scale-true $N(\varphi_m = \pm 41°13.3)$. Shape-true $\lambda=0$; $\varphi = \pm 41°13.3$. F(x) acc. to 123, H semi-ellipses Mollweide (1805) $m = \sqrt{8}:\pi$; $x = \sqrt{2}\sin\psi$, $y = \frac{\sqrt{8}}{\pi}\cos\psi$, where $\pi\sin\varphi = \sin 2\psi + 2\psi$. Scale-true $N(\varphi_m = \pm 45°46')$ Shape-true $\lambda=0$; $\varphi = \pm 45°46'$	TH 39, H165, G184, BF 379; TH 41, ZB 172; (F104), H162, G183, BF 280
136							C F(x) ... from 47 and ... Line-ring maps	Proceed with same x. Nell (1890) x from $2\sin\varphi = x + \sin x$, $2y = \lambda(1+\cos x)$ Scale and shape true on base-line (Special case of 76)	Pet. M.M. 1890, p.93
137								Proceed with same φ. Hammer (1900) $x = 2\varphi$ ..., $2y = \lambda(1+\cos\varphi)$	" " 1900, p.41
138							d F(x) acc to 123	$mn = 4:(2n_1+\pi)$; $n_1 = \cos\varphi'$, x from $m\sin\frac{x}{n} = n$, ... For true scale and shape, $\cos\varphi_m = \frac{\pi}{2}(\cos\frac{x}{m} + n_1)$	
139								Special case of 138. $m = 0.9046$, $n = 1.1054$, $n_1 = 2:\pi$, $mn = 1$. Central H and half base-line 2.8926. Scale-true $N(\varphi_m = \pm 53°0'.5$, ...$)$ True shape at $\lambda=0$, $\varphi = \pm 53°0'.5$	referred to in K.H. Wagner, p.24
140							e F(x) acc to 123, 130	$mn = 4:(3\sqrt{3})$ (.7695, x from $\sin\frac{2x}{3m} = \frac{2}{3m}\sin\varphi$; $y = n\lambda\sqrt{\ldots}$. Generalization of 141. Scale true $N(\pm\varphi_m)$, where $\cos\varphi_m = n:\sqrt{4-3n^2}$; shape-true at $\lambda=0$; $\varphi = \varphi_m$	
141	13							$m = n = \sqrt{1.7695}$ E by K.H. Wagner. Special case of 140. Central H and half base-line equally long. Scale-true $N(\varphi_m = \pm 53°41'.5)$ Shape-true at $\lambda=0$; $\varphi = \pm 53°41'.8$.	
142								$n=1$; $m=0.7698$. Maurer. Special case of 140. Base-line scale and shape-true everywhere	K.H. Wagner p.22 and 23; new
143							f F(x) = parabola $F(x) = \sqrt{3\pi}(1 - \frac{4x^2}{3\pi})$	Craster 1929, $x = \sqrt{3\pi}\sin\frac{\varphi}{3}$; $y = \lambda\sqrt{\frac{3}{\pi}}(1 - \frac{4x^2}{3\pi})$. Scale-true $N(\varphi_m = \pm 36°45')$. Shape true at $\lambda=0$; $\varphi = \pm 36°45'$	Deetz and Adams, p.165
144						B $x = f(\varphi)$ are scribed; $y = \lambda\cos\varphi : \frac{dx}{d\varphi}$	f(φ) proportional to radius according to No 2; $x = m\,tg\frac{\varphi}{2}$	General case. $y = \frac{2\lambda}{m}\cos\varphi\cos^2\frac{\varphi}{2}$; scale true $\varphi = \pm\varphi_m$, where $\cos^2\frac{\varphi_m}{2} = \frac{\pi}{2}$. Shape true at $\lambda=0$; $\varphi = \pm\varphi_m$	new
145								Special case ($m=2$) of 144. Base line scale and shape true	"
146								Répetit-Foucaut (1862) Special case ($m = \sqrt{\pi}$) of 144. Central H and half base-line equally long. Scale-true $H(\varphi_m = \pm 35°32'.5)$. Shape-true at $\lambda=0$; $\varphi = \pm 35°32'.5$	TH 44; H167; G181
147						C Affine transformations from groups 131-143	from 131-133	From 131. All H sine-lines $x_{147} = nx_{131}$; $y = \frac{1}{n}y_{131}$. Example $n = \sqrt{\pi+2}:2 = 1.134$ yields base line scale and shape true	new
148								From 133. All H semi-ellipses; $x_{148} = nx_{133}$; $y = \frac{1}{n}y_{133}$. Example $n = \frac{1}{4}\sqrt{\pi(4+\pi)} = 1.84$ yields baseline scale and shape true	new
149							from group 134-139	From 135. J. Bordin. $x = n\,x_{135}$; $y = \frac{1}{n}y_{135}$. All H semi-ellipses. For $n = \pi:\sqrt{8} = 1.1107$, " " "	TH 40 G190
150	14				b) All N scale-similar $y = n\lambda\cos\varphi$ pointing maps Globe-bangles area-similar	World-map border ellipse of axis ratio 1:2	$x = n\frac{\pi}{2}\sin\varphi$	$n=1$. All N scale-true. Shape-true at $\lambda=0$; $\varphi = \pm 50°27'.6$	
151								$n = \sqrt{8}:\pi$ Globe bangles area-equal. No H scale true. Shape-true $\lambda=0$; $\varphi = \pm 50°27'.6$	
152								$n = 1:\sqrt{\pi}$ Fournier II (1646) The H $\lambda = \pm\frac{\pi}{2}$ are circles. " " . No N scale-true	TH 51

TEXT NOT REPRODUCIBLE

SNo	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature

FAMILY II: STRAIGHT-SYMMETRICAL. BRANCH B: Not secondary-circle-spaced

SNo	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
153 } 154	17	A sing.-maps $f(\pm\varphi)$ finite		All N arcs	For $\lambda^2 \leq \frac{\pi^2}{4}$ N are arcs [illegible] $y = \lambda$ and y c. $x = \pm a$	For $\lambda^2 > \frac{\pi^2}{4}$ N [illegible] points	$x = \varphi$ Central-H scale-true $d = \pi = 2$	$n = 1$ Apianus I. y from $y^2 - y(\frac{\pi^2}{4} - \lambda^2) \cdot \lambda = \frac{\pi^2}{4} - \varphi^2$ for $\lambda^2 < \frac{\pi^2}{4}$; y from $\sqrt{\frac{\pi^2}{4} - \varphi^2} + \lambda - \frac{\pi}{2}$ for $\lambda^2 > \frac{\pi^2}{4}$ Base line scale- and shape-true. Interval-true. $n = 2/3$ " " in the version y from $y^2(\frac{1}{2}\frac{\pi^2}{4} - \frac{1}{3}\lambda^2) : \lambda = \frac{\pi^2}{4} - \varphi^2$ for $\lambda^2 < \frac{9\pi^2}{64}$; $y = \sqrt{\frac{\pi^2}{4} - \varphi^2} + \frac{2\lambda}{3} - \frac{\pi}{2}$ for $\lambda^2 > \frac{9\pi^2}{64}$	TH 97; Kellaway, p. 5; G 25 " "
155	20					$x = \pm a; y = a\lambda \cdot \frac{\pi}{2}$	[illegible]	$n = 1$ Glareanus (1527) Base-line scale and shape-true.	TH 45 G 25
156	20					For $\lambda^2 > \frac{\pi^2}{4}$ as; for $\lambda^2 < \frac{\pi^2}{4}$	[illegible] $\sin \mu = 2\varphi : \pi$	$n = 1$ van der Grinten III Boy line " " " "	ZB 183 (Fig 112)
157	15	B Aims. pr-map $f(\pm\varphi)$	(For further E of this sub-branch see No 215, 216)			$x = n\,tg\,\varphi$	$y = m \sin \frac{\lambda}{2} \cos \varphi$	Special E. N are curves IV order $y \sqrt{n^2 + x^2} = mn \sin \frac{\lambda}{2}$, for $m = \pi$, entire length of all N as on globe; for $n = \varphi_a \cot \varphi_a$ intervals of $N(\varphi = \pm\varphi_a)$ from base-line as on globe.	new

FAMILY III. CURVED-SYMMETRICAL.

N not parallel straight-lines. Grid symmetrical to central-H and to base-line. $x(\lambda/\varphi) = -x(\lambda/\varphi)$; $y(\lambda/\varphi) = -y(-\lambda/\varphi)$; $x(\lambda/0) = y(0/\varphi) = 0$

BRANCH A NON-CIRCULAR. ALL N NOT CONCENTRIC ARCS WITH CENTER-POINTS ON CENTRAL-H. $y^2 + (\Phi + r - x)^2 = r^2$, WHERE r AND Φ ARE FUNCTIONS OF φ. FOR $\varphi = 0$; $\Phi = \frac{1}{r} = 0$

SNo	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
158 } 159 } 160 } 161	72	A Tri-monod. [illegible]	[illegible] maps For $\varphi = \pm 90°$ $y = r = 0$; $\Phi = \pm a$	I. circle-[illegible] of first class [illegible] and λ functions of λ	a) central H scale-true $\Phi = \varphi$; $a = \frac{\pi}{2}$	A. Base-line scale-true. All N arcs through [illegible] ($y = 0$, $x = \pm\frac{\pi}{2}$) and ($x = 0$, $y = \lambda$) [illegible]	a. r determined by dividing points on $\pm H$ $\pm\lambda_0$ and on $\lambda = 0$; r determined by dividing points on $\pm\lambda_0$ and one point $x = 0$, $y = \lambda$	$\lambda_0 = \pm\frac{\pi}{2}$; the 2 H equal-spaced. Globular E by Nicolosi (1660). $2r(\frac{\pi}{2}\sin\varphi - \varphi) = \frac{\pi^2}{4} - \varphi^2 - \pi\varphi\sin\varphi$; $2\varphi(\frac{\pi}{2}\sin\lambda - \lambda) = \frac{\pi^2}{4} - \lambda^2 - \pi\lambda\sin$ [illegible] $\lambda_0 = \pm\pi$, " " " " . Fool-globular E rays of H-lines $\lambda_0 = (\pm\pi) = 5\pi : 5$. Dividing points on these H-lines: $x(\pm\pi) = \frac{5\pi}{4}\sin\varphi$; $y(\pm\pi) = \pm\frac{\pi}{5}(3 + 5\cos\varphi)$. $\lambda_0 = \pm\pi$ Apparently globular E Mercer. Dividing points on H lines λ_0 $x(\pm\pi) = \frac{5\pi}{8}\sin\frac{3\varphi}{4}$; $y = \pm\frac{\pi}{8}(3 + 5\cos\frac{3\varphi}{4})$. $\lambda_0 = \pm\pi$ Van der Grinten's E Dividing points on H-lines λ_0 acc. to equation of these H-lines $x^2 + (y - \frac{3\pi}{4})^2 = (\frac{5\pi}{4})^2$ and equation $y + \frac{\pi}{4} = x\,\frac{\pi - \varphi\sqrt{10}}{\varphi}$	TH 43; ZB 186; H 148; G 253, G 284; Z.d. Ges. f.E. 1922, p. 139, I " " " II " " " III ZB 185 (F114); BF 284
162	72						b. Ar. [illegible] $r = \cot\varphi$	Pseudoconical circle-grid Base line and central-H scale-true. Not rectangular.	
163	72						c. $y^2 + (\Phi + r - x)^2$ [illegible]	$r = \frac{1}{2\varphi}(\frac{\pi^2}{4} - \varphi^2)$ Rectangular circle-grid. Base-line and central-H scale-true. Not pseudoconical	TH 116
164						B Base line not scale-true, cell-equal	H-lines $= \lambda_0$ equal-spaced	$\lambda_0 = \pi$ Dividing points on H lines $\lambda_0 = \pm\pi$: $x(\pm\pi) = 1.673 \sin\frac{2\varphi\alpha_0}{\pi}$; $y(\pm\pi) = \pm 1.673(\cos\frac{2\varphi\alpha_0}{\pi} - \cos\alpha_0)$, where $\alpha_0 = 110°7' = 1.9219$	new
165						$\varphi = \pi(2\sin\alpha)$ [illegible]	b. Zones between the N are cell-equal	Equations in the text.	new
166 } 167 } 168	72			b) central-H not scale-true	A. Base-line scale-true $\Lambda = \lambda$	World map bordered by circle $x = \pi$		$\Phi = \pi\,tg\,\frac{\mu}{2}$, where $\sin\mu = 2\varphi : \pi$; r according to special geometric prescription Van der Grinten I " " " " " " ; $r = \cot\mu$. Rectangular circle-grid. " " " II	ZB 182 (F110); " 183 (111)

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SNo	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
168						B Base-line not scale-true	a. area-equal	Central-H half as long as base-line. Zones between the N area-equal. n = 1.1255 [illegible]	(F 65)
169	67						b shape-true circle-grids $\varrho = n\lambda$	$n = \sqrt{2}$. Lagrange (1797). Change of length ratio in $\varphi = \lambda = 0$ the slowest. Map longer than map plane [Ostwald's Klassiker No 55, p. 56]	TH 123; N 191; G 219, BF 283
170	66						$tg\left[\frac{\pi}{4} - \frac{\varrho}{2}\right] =$	n = 1. All circular E, transverse-axial, map covers map plane exactly	TH 120, N 145; G 228
171							$tg^n\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$	n = 0.7. Base-line about double so long as central-H. Map finite.	new
172	67						$\varrho = a \operatorname{cosec} \alpha$	n = 0.615. Border-meridian-length : equator-length = equator-length : central-H-length = 1.4303 : 1. Map finite.	TH 121
173							$r = a \cot \beta$	n = C.S. Lambert (1772). Base-line just as long as central-H. [Ostwald's Klassiker No 54, p. 74, fig. 11]	ZD 80 (F 103), N 266, G 229
174	72						c. N for $\lambda = \pm 90°$ equal-spaced	Nell's intervening circle-grid Φ and Λ centers from No 158 and 170; $\varrho = (1-\Lambda^2)\cdot 2\Lambda$, where $2\Lambda = \frac{2\lambda}{\pi} + tg\frac{\lambda}{2}$; $2\Phi = \frac{2\varphi}{\pi} + tg\frac{\varphi}{2}$	TH 137, ZD 102, BF 284
175	72			II. Not all N true circles	Central-N scale-true $\Phi = \varphi$	A. Base-line scale-true $\Lambda = \lambda$	a. N for $\lambda = \pm 90°$ equal-spaced semi-circles	Fournier I (1646) H-Lines as in Mollweide (No 135) N-lines circles through the 3 dividing-points on $\lambda = 0, +\frac{\pi}{2}, -\frac{\pi}{2}$. Globe-biangles area-equal	TH 115, N 147, G 256
176	62						b. all-conical (panconical) $r = \cot \varphi$	"Ordinary polyconic E" of American Coast and Geodetic Survey (1855). All N scale-true $\beta = \lambda \sin \varphi$. Shape-true on $\lambda = 0$.	TH 114, ZD 145; (F 90), N 135, G 103, BF 281
177	72							"Rectangular polyconic E" of British War Office (1860) $tg\frac{\beta}{2} = \frac{\lambda}{2}\sin\varphi$. All N and H intersect rectangularly.	TH 111; ZD 148, N 138; G 103, BF 281
178	62						c. Every N-arc thru the 3 points (y = 0, r = φ) and $y = \pm\frac{\pi}{2}$ [illegible] $\frac{\varphi}{\pi}$ sin φ	All N equal spaced Secondary-circle-spaced all-globular. $tg\frac{\beta\pi}{2} = \frac{\varphi \cdot \pi}{y\pi}$; $\beta(\lambda) = \frac{\lambda}{\pi}\beta_\pi$	new
179	64					B. Base-line not scale-true	all-conical (panconical)	Area-true pan-conical E. Meurer β from $\beta = \sin\beta\cos^2\varphi = \lambda\sin^3\varphi$ True-shape at $\varphi = \pm\pi/2$	TH 112, N 146
180	72		A. Pole-line maps for $\varphi = \pm\frac{\pi}{2}$	Circle-grid of first class as 158-174	Central N scale-true	Base-line scale-true $\Lambda = \lambda$	$\varrho = (\pi^2 + \lambda^2) : 2\lambda$	Meurer $r = \pi \cot \mu$, where $\mu = 2a\pi\, tg(\varphi : \pi)$. Grid as 167 but N lines numbered differently.	new
181	67	B. All N thru 2 points $\varphi = 0, \lambda = \pm\pi$; $\varrho = (\pi^2 + \Phi^2) : 2\Phi$	Point-line map	Circle-grid of second class $\varrho^2 = (y - \varrho - \Lambda)^2 = \varrho^2$	[illegible]	Shape-true $\Phi = \operatorname{cosec} B - \cot B$, where $B = n$ lognat $\cot\frac{\xi}{2}$; $\Lambda = \operatorname{cosec}\alpha - \cot\alpha$, where lognat $\cot\frac{\alpha}{2} = n\lambda$		n = 1 Grids as 169-173, but N- and H lines exchanged and should be noted 2λ rather than φ and φ rather than 2λ	TH 127
								FAMILY III. Curved-symmetrical. Branch B - not rowcircular	
182	69	A. Transformed from family I (transverse-axial pole-grid) or from family II or family III branch A	a. Affined E. from a map of E. from which transformed $x_1 = mx$, $y_1 = ny$	I. Transformed from pole-grid of transverse-axial model projections	Transformed from prime point map	A. Original E. plane-perspective.	Original E. center-perspective.	Original E (projection from which transformed) center-perspective (No 1) transverse-axial. Meurer. $x_1 = m\, lg\, \varphi \sec \lambda$, $y_1 = n\, tg\, \lambda$. All great-circles straight-lines. Shape-true at $x_1 = 0$; $y_1 = \pm \sin L$, if $\cos L = n : m$, in $y_1 = 0$; $x_1 = \pm \sin L$, if $\cos L = m : n$ (C.f. Z.f. Vermessungswesen 1922, p. 14)	Pet. M. H. 1919. II, p116
183	64					B. Original E not plane-perspective	Original E area-true	Original E Lambert (No. 23) transverse-axial. Meurer $m = 1.0056$; $x_1 = 2m \sin\frac{\delta}{2}\cos\alpha$; $y_1 = \frac{2}{m}\sin\frac{\delta}{2}\sin\alpha$, where $\cos\delta = \cos\lambda\cos\varphi$ and $tg\,\alpha = \sin\lambda\cot\varphi$; $n = 1 : 1.0056$; shape-true at $\lambda = 0$; $\varphi = \pm 15°$	new
184	64			II. Original E non-circular	Original E. Central-N scale-true	Original E. base-line not scale-true	Original E panconical	Original E. area-true (No. 179) Meurer $n = 1.2743$, $m = 1.2743$; $x_1 = \frac{1}{n}\left(\varphi + 2\cot\varphi\sin^2\frac{\beta}{2}\right)$; $y_1 = n\cot\varphi\sin\beta$; $\beta = \sin\beta\cos^2\varphi = \lambda\sin^3\varphi$. Shape-true at $\varphi = 0$; $\lambda = \pm 46° 19'$	new

SNo	Fhc.	Sub-branch	Order	Class	Subclass	Group	Sort	Type	Literature
155	72		[illegible] $x(\varphi/\lambda)$ and $y(\varphi/\lambda)$ of the orig. E result $x_1(\varphi/2\lambda) = m(\varphi/\lambda)$, $y_1(\varphi/2\lambda) = 2ny(\varphi/\lambda)$	Orig. grid [illegible] transverse axical, radial E.	Orig. E [illegible] map	Orig. E not plane-perspective	Orig. E interval-true	Original E Mercator-Postel No. 24. Aitoff's Planisphere. $m = n = 1$; $x_1(\varphi/2\lambda) = x(\varphi/\lambda)$; $y_1(\varphi/2\lambda) = 2y(\varphi/\lambda)$; where $\cos\sqrt{x^2+y^2} = \cos\lambda\cos\varphi$; $y = x\sin\lambda\cot\varphi$. Base-line and central-H scale-true. Shape-true at $\varphi = \lambda = 0$	ZB 196; BF 250
156	64						Orig. E area-true	Original E Lambert No. 23. Hammer's Planisphere (1892) $m = n = 1:\sqrt{2}$; $x_1(\varphi/2\lambda) = x\frac{(\varphi/\lambda)}{\sqrt{2}}$; $y_1(\varphi/2\lambda) = \sqrt{2}y(\varphi/\lambda)$; $x^2+y^2 = 4\sin^2\frac{z}{2} = 2(1-\cos\varphi\cos\lambda)$; $y = x\sin\lambda\cot\varphi$. Shape-true at $\varphi = \lambda = 0$	ZB 196 (F106); BF 250
157 }	72		C. Meridian-spacing of original E altered	All H equal-spaced	Orig. E circle-grid of first class	Orig. E central-H scale-true $\Phi = \varphi$	Orig. E base-line scale-true $\Lambda = \lambda$	Orig. grid full-globular No 159. Maurer primary-circular-all-globular. All H equal-spaced $\varrho = (\pi^2 + 4\lambda^2):(8\lambda)$, $x = \varrho\sin\frac{2\varphi z}{\pi}$; $y = \lambda + 2\varrho\sin\frac{2\varphi z}{\pi}$, where $\sin z = \pi:(2\varrho)$	Z. d. Ges. f. Erdk. 1922, p. 125
158 }					Orig. E cylinder-circular	Orig. E central-H not equal-spaced	Orig. E area-true	Orig. E Mollweide No. 135. G.G. Schmidt (1803) All H equal-spaced. $x = \sqrt{2}\sin\psi_1$; $y = \frac{2\lambda\sqrt{2}}{\pi}\cos\psi_1$, where $\int_0^{\psi} v\,d\psi$ and $v^2 = 2\cos^2\psi + \frac{8\lambda^2}{\pi^2}\sin^2\psi$. Not area-true	TH 196; H164 footnote
189	65	A Shape-true	N and H algebraic curves.	N and H curves of second order.		N-Ellipses	N-Ellipses confocal	Azimuth-equal map Littrow-Maurer (1833, 1905) H confocal hyperbolae. $x = \operatorname{tg}\varphi\cos\lambda$; $y = \sec\varphi\sin\lambda$. All straight lines on map are azimuth-equals.	TH 151, ZB 221 (F152), H 255
190	67					N-hyperbolae	N-hyperbolae	Fiorini (1881) $x = \sec\lambda\sin\varphi$; $y = \operatorname{tg}\lambda\cos\varphi$	TH 151, H 255
191 }				N and H of higher order		N-curves of fourth order	N-lemniscates	Transformed from 189 by reciprocal rays. $x = \sin\varphi\cos\varphi\cos\lambda : D$, $y = \cos\varphi\sin\lambda : D$, where $D = 1 - \cos^2\lambda\cos^2\varphi$. Azimuth-equals are circles. Maurer	Marine-Luftflotten-Rdschau 1933
192 }	67					H-evolvents of epicycloids		August epicycloidal E (1874) $x_1 + iy_1 = \frac{\rho}{2}(3-\rho^2)$, $\rho = (1+\sqrt{1-Z^2}):Z$; $Z = x + iy$; $x = \sin\varphi : D$, $y = \cos\varphi\sin\lambda : D$, where $D = 1 + \cos\lambda\cos\varphi$. Border meridian an epicycloid.	TH 153
193			N and H are transcendent curves					Eisenlohr (1870) $x + iy = 2i(v + iu) + \sqrt{8}[e^{\dots} - e^{\dots}]$, where $\operatorname{tg} u = \sin\frac{\varphi}{2} : [\cos\frac{\varphi}{2} + \cos\frac{\lambda}{2}\sqrt{2\cos\varphi}$; $v = \frac{1}{2}l\left[\frac{\cos\frac{\varphi}{2} + \sin\dots\cos\varphi\cos(45° - \frac{\lambda}{2})}{\cos\frac{\varphi}{2} + \sqrt{\cos\varphi}\cos(45° + \frac{\lambda}{2})}\right]$	Crelle's Journal, Vol. 72, 1870, p. 143.
194			N and H by double-periodic elliptic integrals of the first class					Guyou (1887) $x = \int_0^{\varphi_e}\frac{d\varphi_e}{\sqrt{1-\frac{1}{2}\sin^2\varphi_e}}$, where $\sin\varphi = \sin\varphi_e\sqrt{1-\frac{1}{2}\sin^2\lambda_e}$; $\cos\varphi\sin\lambda = \sin\lambda_e\sqrt{1-\frac{1}{2}\sin^2\varphi_e}$. Annales hydrographiques, Paris 1887, p. 16	Deetz u. Adams, p. 160, (F. 74)
195	72	C. All H parallel straight lines	Shape-true on base-line and central H	cylinder-radial		double-cylinder-radial	base-line scale-true	Central-H scale-true $\cot x = \cot\varphi\cot\lambda$, $y = \lambda$. All great-circles perpendicular to base-line or to central-H are straight-lines.	Ann. d. Hydr. 1919, p. 20
196	70		Shape-true at two equator-points. (2 further E of this subbranch are special cases of No 199 and 200)			The azimuth-equals of these 2 points are straight-lines		Maurer (1919) double-counter-azimuthal E. Shape-true points $\varphi = 0$; $\lambda = \pm\lambda_1$; $x = 2\sin\varphi(\cos^2\lambda_1 - \sin^2\lambda):\sin 2\lambda_1$; $y = \sin 2\lambda : \sin 2\lambda_1$	

FAMILY IV: Less regular. Neither true-circular nor double-symmetrical

BRANCH A: Grid symmetrical to central-H but not to base-line

SNo	Fhc.	Sub-branch	Order	Class	Subclass	Group	Sort	Type	Literature
197 }	33	A cylinder-radial $y = g(\lambda)$ All N parallel straight lines	a. H straight lines equal-spaced $y = n\lambda$	I. Shape-point maps		a) cylindrical [illegible] A parallel straight lines $x = \frac{1}{n}(\varphi)$; Shape-point below equator-circle $\varphi > 0$	a. intersecting cylindrical	$n = \cos\varphi_m$, $x = \cos\varphi_m[\sin\varphi + \varphi(1-\cos\varphi)]:\cos\varphi$; $y = \lambda\cos\varphi_m$; φ_m scale-true; $\varphi = 0$ shape-true. (For $\varphi = 0$ belonging to family II, Branch A, Subbranch B, No. 42, Fig. 4)	
							b. contiguous-cylindrical	$n = 1$, $x = \operatorname{tg}\varphi + \varphi(1-\cos\varphi):\cos\varphi$, $y = \lambda$. On $\varphi = 0$ scale and shape-true (For $\varphi = 0$ belonging to " " " " ")	

St. No.	F. No.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
199	55				b) A shape-true at S on central [illegible] ($\lambda = 0$; $\varphi = \varphi_0$)	Base line scale-true $n = 1$	A. Great-circles thru S straight lines	Schay's radial-directional E (1913). $y = \lambda$; $x = \lambda(\cos\varphi_0 \, tg\,\varphi - \sin\varphi_0 \, tg\,\lambda) : \sin\lambda$ (For $\varphi_0 = 0$ belonging to family III, branch B, subbranch C, F. No. 77)	Ann. d. Hydr. 1913, p.
200							B. Azimuth-equals at the direction point S are straight lines	Craig's retroazimuthal E (1909) $y = \lambda$, $x = \lambda(\sin\varphi \cos\lambda - \cos\varphi \, tg\,\varphi_0) : \sin\lambda$. Counter-azimuthal. (For $\varphi_0 = 0$ belonging to family III, branch B, subbranch C, F. No. 70)	ZB 213 (F130)
201, 202	75			I. Polar-line maps	All N parallel [illegible] $x = f(\varphi) = \sqrt{r^2 - n^2\lambda^2}$	A. interval-true $f(\varphi) = \varphi$	a. Contiguous cylindrical; b. intersecting cylindrical	$n = 1$, $y = \lambda$, $x = \varphi + r - \sqrt{r^2 - \lambda^2}$. All N greater than circumference of globe. Shape-true at points $\varphi = \lambda = 0$. $n < 1$; $y = n\lambda$, $x = \varphi + r - \sqrt{r^2 - n^2}$. Length of all N = L = $2r\alpha$, where $\sin\alpha = n\pi : r$; $L = 2\pi\cos\varphi_m$ for $\cos\varphi_m = \frac{r\alpha}{\pi} = \frac{n\alpha}{\sin\alpha}$. Shape-true at $\lambda = 0$, $\varphi = \pm\varphi_m$	
203, 204	77					B. area-true $f(\varphi = \sin\varphi)$	a. Contiguous cylindrical; b. intersecting cylindrical	$n = 1$, $y = \lambda$, $x = \sin\varphi + r - \sqrt{r^2 - \lambda^2}$. All N greater than circumference of globe. Shape-true at $\varphi = \lambda = 0$. $n < 1$, $y = n\lambda$; $x = (\sin\varphi + r\sqrt{r^2 - n^2\lambda^2}) : n$. Length of all N = L = $2r\alpha$, where $\sin\alpha = n\pi : r$, $L = 2\pi\cos\varphi_m$ for $\cos\varphi_m = \frac{r\alpha}{\pi} = \frac{n\alpha}{\sin\alpha}$. Shape-true at $\lambda = 0$, $\varphi = \pm\varphi_m$, where $\cos\varphi_m = n$.	
205	73		b. N straight lines not equi-interval	Polar-line maps	All N equi-spaced	All N parallel arcs $y = r\sin\frac{\lambda\alpha}{\pi}$; $x = f(\varphi) + r(1 - \cos\frac{\lambda\alpha}{\pi})$, α and r constant	a. interval-true $f(\varphi) = \varphi$	Type according to values α and r. Scale-true $N (= \varphi_m)$, where $\cos\varphi_m = r\alpha : \pi$; $y = r\sin\frac{\lambda\alpha}{\pi}$; $x = \varphi + r(1 - \cos\frac{\lambda\alpha}{\pi})$	
206	74						b. zone-true	" " " " " " " " , where $\cos\varphi_m = r\alpha : \pi$, $y = \pi\sin\frac{\lambda\alpha}{\pi} : \sin\alpha$; $x = \sin\varphi + 2\pi(\sin\frac{\lambda\alpha}{2\pi})^2 : \sin\alpha$	
207		B. not cylinder radial	a. cylinder-circular $x = f(\varphi)$	I. secondary circle-spaced $y = \lambda g(\varphi)$	A. N straight lines thru point $y = 0$, $x = P$	Point-point maps P_1, namely $y = 0$; $x = P$) picture of a pole	a. Picture of picture of both poles	Loxodrome map. $P = h\csc\beta$, $x = Mh[\frac{1}{h}\cos\beta + M\sin\beta]$; $y = \lambda(1 - \frac{x}{h}\sin\beta)$, β and h constants. $M = \log nat\, tg(\frac{\pi}{4} + \frac{\varphi}{2})$. Perspective of the Mercator map from point h perpendicular over point $\varphi = \lambda = 0$ on plane below angle β at equator.	
208	50						b. Pt. P picture of only one pole	interval-true $P = \frac{\pi}{2}$, $x = \varphi$; $y = \lambda(\pi - 2\varphi) : \pi$ as in No. 105, but same equat. applies for both hemispheres.	
209	53							not interval-true, x and y as in No. 104 and 106, " " " " " " "	
210, 211	50					B. Polar-line maps: no pole shown as point	a. interval-true $P = \frac{\pi}{2}$; $x = \varphi$	$y = \lambda(\pi - \varphi) : \pi$ as in No. 114, " " " " " " " ; $y = \lambda[(\varphi - \varphi_2)\cos\varphi_1 - (\varphi - \varphi_1)\cos\varphi_2] : (\varphi_2 - \varphi_1)$ as in No. 125; " " " " " " " "	
212, 213	51						b. area-true	$x = m\pi - \sqrt{m^2\pi^2 - 2\pi\sin\varphi}$, $y = \lambda(m\pi - x) : \pi$, where $m = \sqrt{3 : \pi}$ as No. 132, but same equat. " " " " ; $x = \sqrt{\pi}[1 - \sqrt{2}\sin(\frac{\pi}{4} - \frac{\varphi}{2})]$, $y = \lambda\sqrt{\frac{2}{\pi}}\sin(\frac{\pi}{4} - \frac{\varphi}{2})$, where $m = 2 : \sqrt{\pi}$ as No. 134, " " " " " "	
214	53						c. neither interval-true nor area-true	x and y as No. 113, 115, 124 and 126, " " " " " "	
215, 216	55, 60			II. not secondary circle spaced	Sec. shape-true point S on central	Polar-point-maps	Azimuth-equals at one direction point S are straight lines	Interval true $x = \varphi - \varphi_0$, $y = -x\, lg\, A$, where $\cot A = (\cos\varphi_0 \, tg\,\varphi - \sin\varphi_0 \cos\lambda) : \sin\lambda$. Counter-azimuthal (For $\varphi_0 = 0$, belongs to fam. II, branch B, Sub. B, F. No. 16) Not interval true, e.g. $x = m\sin\frac{\varphi - \varphi_0}{2}$, $y = -x\, lg\, A$, $A = (\cos\varphi_0$ etc. same as above " " . (" " " " " " " 17)	
217	79		b. not cylinder-circular	I. shape-true	a) d_m straight $\frac{\pi}{2}$	Pow-circular; All N are arcs	Circle-true of the first class	Lambert and Lagrange. $tg(\frac{\pi}{4} - \frac{x}{2}) = (tg\frac{\varphi}{2} : \cot\frac{d_m}{2})^n$; $N r = \cot\beta$, center ($y = 0$, $x = \csc\beta$); [illegible] circle $\varsigma = \csc n\lambda$, center ($x = 0$, $y = -\cot n\lambda$)	TH 129 (F68), H 261
218					b) Go [illegible] symm. to $\lambda = \frac{\pi}{4}$	M and N by double-periodic elliptical integrals of class I.		Peirce (1879) Quincuncial E. Polargrid of the transverse-axial E No. 144. cos at $(x + iy\,k; k) = tg\frac{\varphi}{2}(\cos\lambda + i\sin\lambda)$, where $k = 1 : \sqrt{2}$; $K = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}}$	TH 182, H 267 (F63)
219	76			I. One shape-true pt. S(φ_0) on central M.	Azimuth-equals at direction points are straight lines	Secondary lines of points S drawn to that true-area means		Maurer. Area-true counter azimuthal E. $\frac{x}{y} = tg\,\varphi_0 \cos\varphi \csc\lambda - \sin\varphi \cos\lambda$ (For $\varphi_0 = 0$ belongs to family II, branch B, new subbranch D, File 65)	Pet. Mit. KN II, p. 67

TEXT NOT REPRODUCIBLE

1.

SNo	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
								FAMILY IV. BRANCH B: Grid not symmetrical to central M	
220	79	Pole-in-point maps	All N and M are circles	Circle-grids of first class	Shape-true	North and south poles separated	ocncl	Grid symmetrical to base-line $\varphi = 0$. Shape-true circle grid as No. 171-173; but lateral world map, i.e., $(\lambda + \gamma)$ employed rather than λ.	
221		—	—	—	—			" not " " " " " " " as No. 217.	
222						North and south poles at same point.	n = 0	Grid symmetrical to base line. Obtained from Mercator map by reciprocal rays. Transformation center at $x = 0$, $y = \pi$. All loxodromes are circles.	A. d. Hyd. 1919, p. 219 (r 3)
223								" not " " " " " " " " at $x \gtrless 0$, $y \gtrless \pi$ " " "	
224	40	—	All N and M are straight lines	All loxodromes are straight lines	Cylinder-like	Ascending circle spaced	straight-celled	Mercator map parallel projected onto plane, parallel neither to equator nor to central meridian. M convergent straight lines. $x = l\,g\,n\,at\,tg(\frac{\pi}{4} + \frac{\varphi}{2})$; $y = m\lambda + n x$	
225	52	—		—	Not cylinder-like	Not ascending-circle spaced	Not cellular	" " projected " " " " " " " to central-meridian. Sighting-pt. in the finite. M and N isotegral straight lines.	
								FAMILY V. COMBINATION GRIDS: equations different for different parts of world map.	
								BRANCH A: Double-symmetrical. Equator a straight line	
226		Uniform rule for N	All N parallel straight lines	All M parts of straight lines				Here belong projections listed in family II under No. 104-106, 113-115, 124-126, 132, 134.	
227		Uniform rule for M	All M consist of parts of circle I	All N constructed from arcs	M as in SNo 196			Maurer's Isogonal Map. All N of each four arcs with no breaks. M-equation: $x^2 + [y - tg(\frac{\pi}{4} - \frac{\lambda}{2})]^2 = sec^2(\frac{\pi}{4} - \frac{\lambda}{2})$ N of inner hemisphere $(x - cosec\,\varphi)^2 + y^2 = cot^2\varphi = r_1^2$; 2 connecting arcs with radius $r_2 = tg(\frac{\pi}{4} - \frac{\varphi}{2})$ and closing semi-circle with radius $r_3 = 2r_2$.	Pet. Mitt. 1911, II, p. 91
								BRANCH B: Not double-symmetrical. Equator not a straight-line	
228	—	Uniform rule for M; true-slow lat. r = f (δ)	Star-projections. Northern hemisphere closed. South hemisphere split in n lobes acc. to border meridian. M of south hemisphere are straight lines	M of northern hemisphere are straight lines.	Center-internal-true $\gamma = \delta$	North hemisphere radial	Southern lobes not equal	n = 5 Petermann (1865) Border-meridians 16°, 60°, 100°, 155° E. and 35°, 55°, 120°, 135° W. (Pet. M. Erg., No. 16, p. 67)	TH 144
229	—						Southern lobes equal	n = 5, Berghaus (1879) " " 56°, 125° E.; 16°, 88°, 160° W.	TH 142
230						North hemisphere conical	Southern lobes equal	n = 4 Steinhauser's conoclatic E. Border meridians 0° = 40°, 180°. World map in sector of 240° angle-opening	TH 142
231				M of southern hemisphere are curves	area-true	northern hemisphere radial	southern lobes equal	n = 6. northern-hemisphere as SNo 23, southern hemisphere $\delta > \frac{\pi}{2}$, $r = 2(\sqrt{2} - cos\frac{\delta}{2})$; $a = \lambda cos\frac{\delta}{2} : [\sqrt{2} - cos\frac{\delta}{2}]$, a and λ from straight-line central M. Border-meridians 40°, 100°, 160° E.; 20°, 50°, 140° W.	

TEXT NOT REPRODUCIBLE

S.No	File	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
232.		File not in item no. N co for N	Star-E. North-pole Center-pt. Southern hemisphere slit into n lobes according to straight border meridians.	Equator an n-angle. Each side a symmetry line of the globe-cell halves	Each N a pair of straight-lines, N of each lobe are parallels to their pieces of equator.	All N equal spaced.	Equator irregular n-angle	n = 5. Jäger (1865) Border-meridians as No. 228 (Pet. M. Erg. Heft 16, p. 67)	TH 142
233.						Border-meridian scale-true	Equator regular n-angle.	n=6 Equator and border-meridians: 40°, 100°, 160° E.; 20°, 90°, 140° W	
234						All 4 equal spaced. All N unequal-spaced Area true	Equator regular n-angle	n = 4 For $-\frac{\pi}{4} < \lambda < +\frac{\pi}{4}$ and $\delta < \frac{\pi}{2}$, $y = \sqrt{\pi}\sin\frac{\delta}{2}$, $x = 4\lambda : \sqrt{\pi}$. Neighboring globe-eighths symmetrical to border-line. Border-meridians as 230	
235					Equator a square. Border meridians as diagonals	Equal-area of each globe-eighth	Each N constructed as 4 quadrants and straight-lines	n = 4. In the first quarter of the northern hemisphere Have straight-lines $tg\,\alpha = 4\lambda \cdot \pi$. x_0 and y_0 are independent, for the N, from δ according to equat. $y_0^2 - \left(\frac{\sqrt{\pi}}{2} - x_0\right)^2 = \frac{\pi}{2}$ and $\pi \sin\frac{2\delta}{2} = 2x_0 y_0 \cdot x_0^2 + (y_0 \cdot x_0)^2 \frac{\pi}{4}$. Then, for $x \le x_0$, $y = y_0$ and, for $x > x_0$, $(x - x_0)^2 + (y - x_0) = (y_0 - x_0)^2$.	
236			Perspective projections from a globe onto an n-plane around the globe	Perspective reproduction on each face of the n-plane	Perspective from center of globe	n is small number	regular n-plane	Reproduction on cube, which is contiguous with globe at ($\varphi = 0°$, $\lambda = 10°$ W and 170° E), ($\lambda = 80°$ E, $\varphi = 60°$ N and 30° S), ($\lambda = 100°$ W, $\varphi = 30°$ N and 60° S)	Dietz and Adams p 52, (F 46)
							Irregular n-plane	Reproduction on rectangular eight cornered box. Floor and lid are contiguous with globe at $\varphi = \pm\frac{\pi}{2}$. 4 side-walls parallel to meridian-planes $\lambda = 20°$ E and 70° W, distance $1\frac{1}{2}$ from center of uniform globe. Grid on floor and lid is center-perspective, eith-axical, on sides $y = 1.5\,tg\,\lambda$; $x = 1.5 \sec\lambda\,tg\,\varphi$ figures	new

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13. ABSTRACT

This work is an edited translation of Maurer's Ebene Kugelbilder, Berlin, 1935. It contains an exhaustive classification system for map projection and a catalogue of 237 named projections arranged in classes and various subclasses on the basis of such characteristics as properties (conformality, equivalence, etc.), the appearance of the latitude-longitude grid, and the form of the equations (linear, algebraic transcendental, etc.). It is believed that this translation will help fill a very serious gap in the literature of English-speaking geographers.

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